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19. ABSTRACT (Continue on reverse if necessary and identify by block number)

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$$L(t) = \sum_1 a_i(t,x)^2 \frac{\partial^2}{\partial x_i^2} + b_i(t,x) \frac{\partial}{\partial x_i}.$$

It is shown that the weak solution of a stochastic differential equation

$$dX(t) = dW(t) + L*(t)X(t)dt.$$

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19. Abstract (cont'd).

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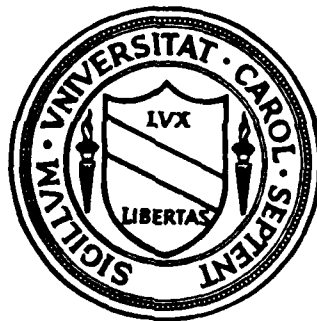
A LANGEVIN-TYPE STOCHASTIC DIFFERENTIAL EQUATION ON A

SPACE OF GENERALIZED FUNCTIONALS

by

G. Kallianpur

and

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205. R.L. Smith. Approximations in extreme value theory. Sept. 87.

A Langevin-type Stochastic Differential Equation on a

Space of Generalized Functionals

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Abstract. Let E' be the dual of a nuclear Fréchet space E and L*(t) the adjoint operator of L(t) which has a formal expression:

$$L(t) = \sum_i a_i(t,x)^2 \frac{\partial^2}{\partial x_i^2} + b_i(t,x) \frac{\partial}{\partial x_i}.$$

It is shown that the weak solution of a stochastic differential equation:

$$dX(t) = dW(t) + L*(t)X(t)dt,$$

exists uniquely on a generalized functional space on E' which is an appropriate model for the central limit theorem for an interacting system of spatially extended neurons. Applications to the latter problem are discussed.

Key words and phrases: Weak solution, SDE, Fréchet derivative, generalized functional space, central limit theorem, system of neurons.

§1. Introduction

Recently, Deuschel [4] has obtained a fluctuation result for a system of lattice valued diffusion processes. The result obtained is similar to the ones for mean-field interacting particle diffusions treated in a number of papers , [2,3,8,9,16,23]. In another direction, Kallianpur and Wolpert [11] have introduced a class of stochastic differential equations (SDE's) governing nuclear space valued processes as a model for voltage potentials for spatially extended neurons. The present paper is motivated by both the above problems, especially, the problem of interacting systems of neurons. The techniques developed in this paper enable us to prove a general result which yields a central limit theorem for such systems. It also provides another approach to the fluctuation theorem in [4]. In addition, the identification problem of the limit measures leads us to discuss the uniqueness of weak solutions of the SDE , formally expressed by

$$dX(t) = dW(t) + L^*(t)X(t)dt.$$

A precise meaning to the above equation is given by equation (1.1) below.

Our aim is to find a suitable space $\mathcal{D}_E$, of smooth functionals on the dual nuclear space E' and to solve the SDE on the dual space $\mathcal{D}_E'$,, which is appropriate for the central limit theorems we have in mind.

We will proceed to explain the setting: A stochastic process $X_F(t)$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$ indexed by elements in $\mathcal{D}_E$, is called an $\mathcal{L}(\mathcal{D}_E)$-process if $X_F(t)$ is a real stochastic process for any fixed $F \in \mathcal{D}_E$, and $X_{\alpha F + \beta G}(t) = \alpha X_F(t) + \beta X_G(t)$ almost surely for each real numbers α, β and elements of $F, G \in \mathcal{D}_E$, and further $E[X_F(t)^2]$ is continuous with respect to F on $\mathcal{D}_E$. [10]. $X_F(t)$ is called continuous if $\lim_{t \to s} E[(X_F(t) - X_F(s))^2] = 0$ for each $F \in \mathcal{D}_E$,. Let $W_F(t)$ be an $\mathcal{L}(\mathcal{D}_E)$-Wiener process, i.e. such that for any fixed

$F \in \mathcal{D}_{E'}$, $W_F(t)$ is a real continuous Gaussian additive process with mean 0.

We will prove that a unique continuous $\mathcal{L}(\mathcal{D}_{E'})$-process solution $X_F(t)$ exists for the following equation with given initial value $X_F(0)$:

(1.1)
$$dX_F(t) = dW_F(t) + X_{L(t)F}(t)dt.$$

Roughly speaking, if $L(t)$ generates the strongly continuous Kolmogorov evolution operator $U(t,s)$ from $\mathcal{D}_{E'}$ into itself, the unique solution for (1.1) can be given as follows:

$$X_F(t) = X_{U(t,0)F}(0) + W_F(t) + \int_0^t W_{L(s)U(t,s)F}(s)ds.$$

We will now begin by giving the precise definitions of the operator $L(t)$ and the space $\mathcal{D}_{E'}$. Let E be a nuclear Fréchet space whose topology is defined by an increasing sequence of Hilbertian semi-norms $\|\cdot\|_1 \leq \|\cdot\|_2 \leq \cdots \leq \|\cdot\|_p \cdots$. As usual let E' be the dual space, E_p the completion of E by the p-th semi-norm $\|\cdot\|_p$ and E_p' the dual space of E_p. Then we have

$$E = \bigcap_{p=0}^{\infty} E_p \quad \text{and} \quad E' = \bigcup_{p=0}^{\infty} E_p'.$$

Let K be a separable Hilbert space with norm $\|\cdot\|_K$ and F a mapping from E' into K. Then F is said to be E_p'-Fréchet differentiable if for every $x \in E'$, we have a bounded linear operator $\mathcal{D}_p F(x)$ from E_p' into K such that

$$\lim_{t \to 0} \frac{F(x+th) - F(x)}{t} = \mathcal{D}_p F(x)(h), \quad h \in E_p', \text{ in K}.$$

Suppose that F is E_p'-Fréchet differentiable for every integer $p \geq 0$. Then taking $E' = \bigcup_{p=0}^{\infty} E_p'$ and the strong topology of E', (which is equivalent to the inductive limit topology of E_p'; $p=0,1,2,\cdots$), into account, we have a continuous linear operator $DF(x)$ from E' equipped with the strong topology into

K such that for any integer $p \geq 0$, $DF(x)(h) = \mathfrak{D}_p F(x)(h)$ for $h \in E_p'$. Hence, if F is n-times E_p'-Fréchet differentiable for every integer $p \geq 0$, we have a continuous n-linear operator $D^n F(x)$ from $\underbrace{E' \times E' \times \cdots \times E'}_{n\text{-times}}$ into K such that the restriction of $D^n F(x)$ on $\underbrace{E_p' \times E_p' \times \cdots \times E_p'}_{n\text{-times}} =$ the n-th E_p'-Fréchet derivative $\mathfrak{D}_p^n F(x)$.

Then if F is infinitely many times E_p'-Fréchet differentiable for every integer $p \geq 0$, the Hilbert-Schmidt norm

$$\|D^n F(x)\|_{H.S.}^{(p)} = \left(\sum_{i_1, i_2, \cdots, i_n = 1}^{\infty} \|D^n F(x)(h_{i_1}^{(p)}, h_{i_2}^{(p)}, \cdots, h_{i_n}^{(p)})\|_K^2 \right)^{1/2}$$

is finite for each integer $n \geq 1$ and $p \geq 0$, where $(h_j^{(p)})$ is a C.O.N.S., (complete orthonormal system), in E_p' [14].

From now on, we will often use the conventional notation such that $\|D^0 F(x)\|_{H.S.}^{(p)} = \|F(x)\|_K$.

Let $\beta(t)$ be the standard E'-Wiener process such that for any $\xi \in E$, $\langle \beta(t), \xi \rangle$ is a 1-dimensional Brownian motion, with variance $E[\langle \beta(t), \xi \rangle^2] = t\|\xi\|_0^2$, where $\langle x, \xi \rangle$, $(x \in E'$, $\xi \in E)$, denotes the canonical bilinear form on $E' \times E$.

Without loss of generality, we assume $\beta(t)$ is an E_1'-valued Wiener process throughout this paper, [17], [18].

Definition of L(t). For $t > 0$ and $x \in E'$, let $A(t,x)$ and $B(t,\cdot)$ be continuous mappings from E' into itself such that the following conditions are satisfied.

(H1) There exists a natural number p_0 such that $A(t,x)$ maps E_1' into E_{p_0}'. $B(t,\cdot)$ maps E' into E_{p_0}' and for each $T > 0$,

$$\sup_{\substack{x \in E' \\ 0 \leq t \leq T}} | A(t,x) |_2^2 < \infty \quad \text{and} \quad \sup_{\substack{x \in E' \\ 0 \leq t \leq T}} \|B(t,x)\|_{-p_0} < \infty,$$

where $\|\cdot\|_{-p}$ denotes the dual norm of E_p' and $| A(t,x) |_2^2 = \sum_{j=1}^{\infty} \|A(t,x)h_j^{(0)}\|_{-p_0}^2$.

(H2) $A(t,x)$ and $B(t,x)$ are infinitely many times E_p'-Fréchet differentiable for every integer $p \geq 0$ such that for any $T > 0$ and any integer $n \geq 1$,

$$\sup_{\substack{x \in E' \\ 0 \leq t \leq T}} \|D^n(t,x)\|_{H.S.}^{(p)} < \infty \quad \text{and} \quad \sup_{\substack{x \in E' \\ 0 \leq t \leq T}} \|D^n B(t,x)\|_{H.S.}^{(p)} < \infty,$$

where $\|D^n A(t,x)\|_{H.S.}^{(p)} = (\sum_{i_1, i_2, \cdots, i_n = 1}^{\infty} | D^n A(t,x)(h_{i_1}^{(p)}, h_{i_2}^{(p)}, \cdots, h_{i_n}^{(p)}) |_2^2)^{1/2}$ and

$\|D^n B(t,x)\|_{H.S.}^{(p)} = (\sum_{i_1, i_2, \cdots, i_n = 1}^{\infty} \|D^n B(t,x)(h_{i_1}^{(p)}, h_{i_2}^{(p)}, \cdots, h_{i_n}^{(p)})\|_{-p_0}^2)^{1/2}$.

(H3) For any integer $n \geq 0$ and any $T > 0$, there exist $\lambda(n,P,T) > 0$ and $\lambda_1(n,p,T) > 0$ such that

$$\sup_{\substack{x \in E' \\ 0 \leq k \leq n}} \max\{\|D^k A(t,x) - D^k A(s,x)\|_{H.S.}^{(p)}, \|D^k B(t,x) - D^k B(s,x)\|_{H.S.}^{p}\}$$

$$\leq \lambda_1(n,p,T) \, |t-s|^{\lambda(n,P,T)}, \quad 0 \leq s, t \leq T.$$

Then for any twice E_p'-Fréchet differentiable real valued functional F on E' for every $p \geq 0$, we put

$$(L(t)F)(x) = \frac{1}{2} \text{ trace }_{E_0} D^2 F(x) \circ [A(t,x) \times A(t,x)] + DF(x)(B(t,x)),$$

where $\text{ trace }_{E_0} D^2 F(x) \circ [A(t,x) \times A(t,x)] = \sum_{j=1}^{\infty} D^2 F(x)(A(t,x)h_j^{(0)}, A(t,x)h_j^{(0)})$.

Definition of $\mathfrak{I}_{E'}$. For a real valued infinitely many times E_p'-Fréchet differentiable functional F on E' for every integer $p \geq 0$, we define the following semi-norms:

$$\|F\|_{p,q,n} = \sum_{k=0}^{n} \|F\|_{p,k}^{(q)} .$$

where $p \geq 0$, $q \geq 0$ and $n \geq 0$ are integers and

$$\|F\|_{p,k}^{(q)} = \sup_{x \in E_p'} e^{-\|x\|-p} \|D^k F(x)\|_{H.S.}^{(q)} .$$

For any natural number n, define

$$S(\mathbb{R}^n) = \{\phi(x) = h(x)\varphi(x);\ \varphi \in \mathcal{Y}(\mathbb{R}^n)\}.$$

where $\varphi(x)$ is an element of the Schwartz space $\mathcal{Y}(\mathbb{R}^n)$ of rapidly decreasing C^∞-functions on the n-dimensional Euclidean space $\mathbb{R}^n$ and $h(x)$, $x = (x_1, x_2, \ldots, x_n)$, is a weight function such that $h(x) = 1/g(x)$,

$g(x) = \prod_{i=1}^{n} g_0(x_i)$, $g_0(x_i) = \exp(-\sqrt{\int_{\mathbb{R}} |y| \rho(x_i - y) dy})$ and $\rho(x)$ is the Friedrichs mollifier whose support is contained in $[-1,1]$. Let $\{\xi_j;\ j=1,2,\ldots\}$ be a countable dense subset of E. Define

$$C_{0,n}^{\infty}(E') = \{\phi(x) = \phi(\langle x, \xi_1 \rangle, \langle x, \xi_2 \rangle, \cdots, \langle x, \xi_n \rangle);\ \phi \in S(\mathbb{R}^n)\}$$

and introduce the nuclear Fréchet topology on this space by the countably many semi-norms;

$$\|\phi\|_p = \sup_{\substack{x \in \mathbb{R}^n \\ 0 \leq k \leq p}} (1+|x|^2)^p \left| \left(\frac{d}{dx}\right)^k (g(x)\phi(x)) \right|, \quad p=0,1,2,\ldots,$$

where $\left(\frac{d}{dx}\right)^k = \sum_{k_1+k_2+\ldots k_n=k} \dfrac{\partial^k}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_n^{k_n}}$. Then we have a fundamental space $C_0^{\infty}(E') = \bigcup_{n=1}^{\infty} C_{0,n}^{\infty}(E')$ which is the strict inductive limit of nuclear Fréchet spaces $C_{0,n}^{\infty}(E')$.

For any integers $p \geq 0$, $q \geq 0$ and $n \geq 0$, let $\mathcal{D}_{p,q,n}$ be the completion of $C_0^{\infty}(E')$ by the semi-norm $\|\cdot\|_{p,q,n}$. We define $\mathcal{D}_{E'} = \bigcap_{p,q,n} \mathcal{D}_{p,q,n}$ and introduce a

topology on $\mathfrak{D}_{E'}$ by the countably many semi-norms $\|\cdot\|_{p,q,n}$, $p \geq 0$, $q \geq 0$ and $n \geq 0$.

Then $\mathfrak{D}_{E'}$ becomes a complete separable metric space [6].

Remark 1. The definition of $\mathfrak{D}_{E'}$ is independent of the way of choosing a countable dense subset of E. We call a real valued functional expressed as $\Phi(x) = \phi(\langle x, \xi_1 \rangle, \langle x, \xi_2 \rangle, \cdots, \langle x, \xi_n \rangle)$ by using some natural number n, $\xi_i \in E$, $i = 1, 2, \cdots, n$, and $\phi \in S(\mathbb{R}^n)$ a weighted Schwartz functional. Let $\mathscr{S}$ be the set of all weighted Schwartz functionals, $\mathscr{S}_{p,q,n}$ the completion of $\mathscr{S}$ by $\|\cdot\|_{p,q,n}$ and $\mathfrak{D} = \bigcap_{p,q,n} \mathscr{S}_{p,q,n}$, where $p \geq 0$, $q \geq 0$ and $n \geq 0$ are integers. Then

$$\mathfrak{D} = \mathfrak{D}_{E'}.$$

Proof. It is enough to show that $\Phi(x) = \Phi(\langle x, \xi_1 \rangle, \langle x_1, \xi_2 \rangle, \ldots, \langle x, \xi_m \rangle)$, $\xi_i \in E$, $\phi \in S(\mathbb{R}^m) \in \mathfrak{D}_{p,q,n}$. By the nuclearity of E, we have a natural number $r > \max\{p,q\}$ such that

$$(1.2) \qquad \sum_{j=1}^{\infty} \|h_j^{(q)}\|_{-r}^2 < \infty$$

and since $\{\overset{o}{\xi}_j\}$ is dense in E, for each i, there exists a sequence $\{\overset{o}{\xi}_{i,k}\}$, $\overset{o}{\xi}_{i,k} \in \{\overset{o}{\xi}_j\}$ such that

$$(1.3) \qquad \lim_{k \to \infty} \|\xi_i - \overset{o}{\xi}_{i,k}\|_r = 0.$$

On the other hand, $D^n \Phi(x)(h_{i_1}^{(q)}, h_{i_2}^{(q)}, \cdots, h_{i_n}^{(q)})$ is a finite sum of terms;

$$(1.4) \qquad \frac{\partial^n}{\partial x_1^{n_1} \partial x_2^{n_2} \cdots \partial x_m^{n_m}} \phi(\langle x, \xi_1 \rangle, \langle x, \xi_2 \rangle, \cdots, \langle x, \xi_m \rangle) \langle h_{j_1^{(1)}}^{(q)}, \xi_1 \rangle$$

$$\times \langle h_{j_2^{(1)}}^{(q)}, \xi_1 \rangle \cdots \langle h_{j_{n_1}^{(1)}}^{(q)}, \xi_1 \rangle \langle h_{j_1^{(2)}}^{(q)}, \xi_2 \rangle \langle h_{j_2^{(2)}}^{(q)}, \xi_2 \rangle \cdots \langle h_{j_{n_2}^{(2)}}^{(q)}, \xi_2 \rangle$$

$$\cdots \; \langle h^{(q)}_{j_1(m)}, \xi_m \rangle \; \langle h^{(q)}_{j_2(m)}, \xi_m \rangle \; \cdots \; \langle h^{(q)}_{j_{n_m}(m)}, \xi_m \rangle.$$

where $n_1 + n_2 + \cdots + n_m = n$. Since

$$(1.5) \qquad \left| \frac{\partial^\mu}{\partial x_1^{\mu_1} \partial x_2^{\mu_2} \cdots \partial x_m^{\mu_m}} h(x) \right| \le C_1 \exp \left(\sum_{i=1}^{m} \sqrt{|x_i|} \right).$$

Noticing $\phi(x) = h(x)\varphi(x)$, $\varphi \in \mathcal{S}(\mathbb{R}^m)$, we have

$$(1.6) \quad \sup_k \sup_{x \in E_p'} e^{-\|x\|_{-p}} \max \left\{ \left| \frac{\partial^n}{\partial x_1^{n_1} \partial x_2^{n_2} \cdots \partial x_m^{n_m}} \phi(\langle x, \xi_1 \rangle, \langle x, \xi_2 \rangle, \cdots, \langle x, \xi_m \rangle) \right|, \right.$$

$$\left| \frac{\partial^n}{\partial x_1^{n_1} \partial x_2^{n_2} \cdots \partial x_m^{n_m}} \phi(\langle x, \xi_{1,k} \rangle, \langle x, \xi_{2,k} \rangle, \cdots, \langle x, \xi_{m,k} \rangle) \right|,$$

$$\left| \frac{\partial^n}{\partial x_1^{n_1} \cdots \partial x_{i-1}^{n_{i-1}} \partial x_i^{n_i+1} \partial x_{i+1}^{n_{i+1}} \cdots \partial x_m^{n_m}} \phi(\langle x, \xi_{1,k} \rangle, \langle x, \xi_{2,k} \rangle, \cdots \right.$$

$$\cdots, \langle x, \xi_{i,k} \rangle + \tau(\langle x, \xi_i \rangle - \langle x, \xi_{i,k} \rangle), \langle x, \xi_{i+1} \rangle, \cdots, \langle x, \xi_m \rangle) \Big|$$

$$\times \|x\|_{-p}, \; 0 \le \tau \le 1, \; i = 1, 2, \cdots, m \right\} \le C_2.$$

Setting $\psi^{(k)}(x) = \phi(\langle x, \xi_{1,k} \rangle, \langle x, \xi_{2,k} \rangle, \cdots, \langle x, \xi_{m,k} \rangle)$ and using (1.2) – (1.6), we have

$$\lim_{k \to \infty} \| \phi - \psi^{(k)} \|_{p,q,n} = 0,$$

which completes the proof.

Here and in the sequel, we denote positive constants by C_i or, if necessary, by $C_i(\tau_1, \tau_2, \cdots)$, $i = 1, 2, \cdots$, in case they depend on the parameters $\tau_1, \tau_2, \cdots$.

Before proceeding to the discussion of equation (1.1), the following remarks on the $\mathcal{L}(\mathfrak{D}_{E'})$-Wiener process are in order. Taking the continuity of $W_F(t)$ and $E[W_F(t)^2]$ with respect to the parameters t and F into account, we note that $\sup_{0 \leq t \leq T} E[W_F(t)^2] < \infty$ and $\sup_{0 \leq t \leq T} E[W_F(t)^2]$ is lower semi-continuous on $\mathfrak{D}_{E'}$. Since $\mathfrak{D}_{E'}$ is a complete metric space, by the Banach-Steinhaus theorem we have some positive integers p_1, q_1 and m_1 such that

$$(1.7) \qquad \sup_{0 \leq t \leq T} E[W_F(t)^2] \leq C_3(T) \|F\|^2_{p_1, q_1, m_1}.$$

Now given a functional $V_t(F)$ such that for each t it is a positive definite quadratic form on $\mathfrak{D}_{E'} \times \mathfrak{D}_{E'}$, increasing and continuous in t and $\sup_{0 \leq t \leq T} V_t(F) \leq C_4(T) \|F\|^2_{p,q,n}$ for some natural numbers p, q and n, we can construct a $\mathfrak{D}_{E'}$-indexed Gaussian mean-zero continuous process $W_F(t)$ with independent increments and variance $V_t(F)$ by the Kolmogorov theorem, since $V_{t \wedge s}(F)$ is positive definite quadratic form with respect to (t,F), $t \in [0,\infty)$, $F \in \mathfrak{D}_{E'}$. Here $t \wedge s = \min\{t,s\}$.

§2. Existence and Uniqueness of solutions of the SDE

Let $\eta_{s,t}(x)$ be a solution of the following stochastic differential equation:

$$\eta_{s,t}(x) = x + \int_s^t A(r, \eta_{s,r}(x)) d\beta(r) + \int_s^t B(r, \eta_{s,r}(x)) dr,$$

where $\beta(t)$ is the standard E'-Wiener process. By the assumptions (H1) and (H2), if $p \geq p_0$ and $x \in E'_p$, then the solution of the above equation is uniquely obtained by the usual method of successive approximations in E'_p.

We will assume the following condition:

(H4) $(L(t)F)(x)$ and $(U(t,s)F)(x) = E[F(\eta_{s,t}(x))] \in \mathfrak{D}_{E'}$ if $F \in \mathfrak{D}_{E'}$.

Let $W_F(t)$, $F \in \mathfrak{D}_{E'}$ be the $\mathcal{L}(\mathfrak{D}_{E'})$-Wiener process and L(t) the diffusion

operator defined above. Then we will prove

Proposition 1. Under the assumptions (H1)-(H4) the continuous $\mathcal{L}(\mathfrak{D}_E.)$-process solution of (1.1) such that for some $0<\alpha<1$, $E[\,|X_F(0)|^{2+\alpha}\,] < \infty$ is uniquely given as follows:

$$X_F(t) = X_{U(t,0)F}(0) + W_F(t) + \int_0^t W_{L(s)U(t,s)F}(s)ds.$$

Proof. Under the assumptions (H1)-(H4), $L(t)$ is a continuous linear operator from $\mathfrak{D}_E.$ into itself. We use the following lemma which will be proved later.

Lemma 1. Suppose that the conditions (H1)-(H4) hold. Then $L(t)$ generates the Kolmogorov evolution operator $U(t,s)$ from $\mathfrak{D}_E.$ into itself such that

(1) $U(t,s)$ is a continuous linear operator from $\mathfrak{D}_E.$ into itslef,

(2) for any $F \in \mathfrak{D}_E.$, $U(t,s)F$ is continuous from $\{(t,s); \ 0{\leq}s{\leq}t\}$ into $\mathfrak{D}_E..$

(3) $U(t,t) = U(s,s) =$ identity operator,

(4) $\dfrac{d}{dt} U(t,s)F = U(t,s)L(t)F, \ 0 \leq s \leq t$ on $\mathfrak{D}_E..$

(5) $\dfrac{d}{ds} U(t,s)F = -L(s)U(t,s)F, \ 0{\leq}s{\leq}t, \ t{>}0$ on $\mathfrak{D}_E.$.

Further for any integers $p{\geq}p_0$, $q{\geq}0$, $n{\geq}0$, $j{\geq}1$ and any $T{>}0$ and $F \in \mathfrak{D}_E..$, we have

(2.1) $\|U(.',s')F - U(t,s)F\|_{p,q,n}^{2j} \leq C_5(T,F,p,q,n)\{|t-t'|^j + |s-s'|^j\}$,
$$0 \leq s,t,s',t' \leq T.$$

First we will verify that the integral in Proposition 1 is well defined by showing that for any fixed $F \in \mathfrak{D}_E..$, $W_{L(s)U(t,s)F}(s)$ is continuous in (t,s). Since $W_F(t)$ is a Gaussian additive process with mean 0 and variance $V_t(F)$, we get for any integer $n \geq 1$,

$$(2.2) \quad E[|W_F(t_1) - W_F(t_2)|^{2n}] \leq C_6(T)(V_{t_1}(F) - V_{t_2}(F))^n, \quad 0 \leq t_1, t_2 \leq T.$$

We choose an integer $k \geq 4$ such that $2k\lambda(m_1, q_1, T) > 2$, where m_1 and q_1 are the numbers which appeared in (1.7) and $\lambda(m_1, q_1, T)$ is the number in (H3). For $0 \leq s, t, s', t' \leq T$, the inequalities (2.1) and (2.2) yield, together with (H3),

$$(2.3) \qquad E\{|W_{L(s)U(t,s)F}(s') - W_{L(s)U(t,s)F}(s)|^{2k}]$$

$$\leq C_7(T)(V_{s'}(L(s)U(t,s)F) - V_s(L(s)U(t,s)F))^k$$

and

$$(2.4) \qquad E[|W_{L(s')U(t',s')F}(s') - W_{L(s)U(t,s)F}(s')|^{2k}]$$

$$\leq C_8(T)\|L(s')U(t',s')F - L(s)U(t,s)F\|^{2k}_{p_1, q_1, m_1}$$

$$\leq C_9(T)\{\|U(t',s')F - U(t,s)F\|^{2k}_{p_1, q_1, m_1+1} + \|U(t',s')F - U(t,s)F\|^{2k}_{p_1, q_1, m_1+2}$$

$$+ |s'-s|^{2k\lambda(m_1, q_1, T)}\}$$

$$\leq C_{10}(T)\{|t-t'|^k + |s-s'|^k + |s'-s|^{2k\lambda(m_1, q_1, T)}\}.$$

The inequalities (2.3) and (2.4) are sufficient for the Kolmogorov-Totoki criterion [25] for continuity in (t,s). The continuity of $W_{L(s)U(t,s)L(t)F}(s)$ in (t,s) can be proved similarly.

Now we proceed to the proof of the existence of solutions for (1.1). Taking the relation $U(t,s)F = F + \int_s^t U(\tau,s)L(\tau)Fd\tau$, the continuity of $W_{L(s)U(\tau,s)L(\tau)F}(s)$ in τ, the linearity of $W_*(s)$ and the L^2-continuity of $W_*(s)$, into account, we have

$$W_{L(s)U(t,s)F}(s) = W_{L(s)F}(s) + W_{L(s)\int_s^t U(\tau,s)L(\tau)Fd\tau}(s)$$

$$= W_{L(s)F}(s) + \int_s^t W_{L(s)U(\tau,s)L(\tau)F}(s)d\tau.$$

so that by making use of the continuity of $W_{L(s)U(\tau,s)L(\tau)F}(s)$ in (τ,s) again, we get

(2.5)
$$\int_0^t W_{L(s)U(t,s)F}(s)ds$$

$$= \int_0^t W_{L(s)F}(s)ds + \int_0^t (\int_s^t W_{L(s)U(\tau,s)L(\tau)F}(s)d\tau)ds$$

$$= \int_0^t W_{L(s)F}(s)ds + \int_0^t (\int_0^\tau W_{L(s)U(\tau,s)L(\tau)F}(s)ds)d\tau$$

$$= \int_0^t (W_{L(\tau)F}(\tau) + \int_0^t W_{L(s)U(\tau,s)L(\tau)F}(s)ds)d\tau$$

$$= \int_0^t (X_{L(\tau)F}(\tau) - X_{U(\tau,0)L(\tau)F}(0))d\tau.$$

Combining the L^2-continuity of $X_F(0)$ in the definition of $\mathcal{L}(\mathfrak{D}_E.)$-process and the Jensen inequality such that $E[|X_F(0)|^{2+\alpha}] \leq E[|X_F(0)|^2]^\alpha$, we get that $E[|X_F(0)|^{2+\alpha}]$ is continuous in $\mathfrak{D}_E.$. Hence there exist positive integers $P_2 \geq P_0.q_2$ and m_2 such that

(2.6)
$$E[|X_F(0)|^{2+\alpha}] \leq C_{11}\|F\|_{P_2.q_2.m_2}^{2+\alpha}.$$

Therefore the Kolmogorov criterion for continuity, together with the inequalities (2.1) in Lemma 1 and (2.6), yields the continuity of $X_{U(\tau,0)L(\tau)F}(0)$ in τ. Thus we get

(2.7)
$$\int_0^t X_{U(\tau,0)L(\tau)F}(0)d\tau = X_{U(t,0)F}(0) - X_F(0).$$

The equalities (2.5) and (2.7) show that $X_F(t)$ is a solution of the equation (1.1).

Following H. Komatsu [12], we now prove the uniqueness of L^2-continuous solutions for the equation (1.1). Let $Y_1(t,F)$ and $Y_2(t,F)$ be the two continuous $\mathcal{L}(\mathfrak{D}_E.)$-process solutions for the equation (1.1). First we remark by the Baire category theorem that for each $T > 0$, we have some natural number

$p_3 \geq p_0 \cdot q_3$ and m_3 such that

$$(2.8) \qquad \max_{i=1,2} \sup_{0 \leq t \leq T} E[Y_i(t,F)^2] \leq C_{12}(T)\|F\|_{p_3 \cdot q_3 \cdot m_3}.$$

Define $v(t,F) = Y_1(t,F) - Y_2(t,F)$. Then for any $a > 0$, we will prove $\frac{d}{dt} E[v(t,U(a,t)F)^2] = 0$ for $t \in (0,a]$. The inequality (2.8) and the strong continuity of $U(t,s)$, ((2) in Lemma 1), yield

$$E[|\frac{v(s,U(a,s)F)^2 - v(t,U(a,t)F)^2}{s - t}|]$$

$$\leq C_{13}(T,F) \, E[(\frac{v(s,U(a,s)F)-v(t,U(a,t)F)}{s - t})^2]^{1/2}, \quad s,t \in (0,a] \subset [0,T].$$

The inequality (2.8) and the strong continuity of $L(t)$ and $U(t,s)$ imply that

$$(2.9) \qquad \lim_{s \to t} E[|\frac{v(s,U(a,t)F) - v(t,U(a,t)F)}{s - t} - v(t,L(t)U(a,t)F)|^2] = 0.$$

By the strong continuity of $U(t,s)$, we get similarly

$$(2.10) \qquad \lim_{s \to t} E[|\frac{v(s,[U(a,s) - U(a,t)]F) - v(t,[U(a,s) - U(a,t)]F)}{s - t}$$

$$- v(t,L(t)[U(a,s) - U(a,t)]F)|^2] = 0.$$

Since $L(t)$ generates the Kolmogorov evolution operator $U(t,s)$, we have

$$\lim_{s \to t} E[|v(t,L(t)U(a,s)F) - v(t,L(t)U(a,t)F)|^2] = 0$$

$$\lim_{s \to t} E[|v(t,L(t)U(a,t)F) + v(t,\frac{U(a,s) - U(a,t)}{s - t} F)|^2] = 0,$$

so that we get

$$(2.11) \qquad \lim_{s \to t} E[|v(t,L(t)U(a,s)F) + \frac{v(t,U(a,s)F) - v(t,U(a,t)F)}{s - t}|^2] = 0.$$

From (2.9), (2.10) and (2.11), we get the desired equality claimed above.

Hence $E[v(t,U(a,t)F)^2] = $ constant. Then letting $t \to 0$, by (2.8) and the definition of continuity of an $\mathcal{L}(\mathfrak{D}_E.)$-process in t, we have the constant $= 0$. Taking the equalities $E[v(t,U(a,t)F)^2] = E[(v(t,F) + v(t,[U(a,t) - U(a,a)]F))^2]$ and $\lim_{t \to a} E[v(t,[U(a,t) - U(a,a)]F)^2] = 0$, into account, we have $E[v(a,F)^2] = 0$ for any $a > 0$, which implies $v(a,F) = 0$ almost surely. Thus the proof is completed.

§3. Proof of Lemma 1.

As in [20], [21], we will treat the generation problem via the stochastic method.

For any F in $\mathfrak{D}_E.$, we recall the definition of $U(t,s)$:

$$(U(t,s)F)(x) = E[F(\eta_{s,t}(x))].$$

To examine that $U(t,s)$ is the evolution operator stated in Lemma 1, we will check some regularities and integrabilities for $\eta_{s,t}(x)$. It is obvious that if $p \geq p_0$ and $x \in E_p'$, $\eta_{s,t}(x) \in E_p'$, so that for $h \in E_{p_4}'$, $\eta_{s,t}(x+h) \in E_{p_5}'$, where $p_5 = p \vee p_4$. Here $a \vee b = \max\{a,b\}$. Following Kunita (p. 219 of [13]), we will show that $\xi_{s,t}(\tau) := \frac{1}{\tau}\{\eta_{s,t}(x+\tau h) - \eta_{s,t}(x)\}$ has a continuous extension at $\tau = 0$ for any s,t a.s. in E_{p_5}'. This can be proved by appealing to the Kolmogorov-Totoki criterion for continuity [25].

Lemma 2. For any $T>0$ and any integer $j \geq 1$, we have

$$E[\|\xi_{s,t}(\tau) - \xi_{s',t'}(\tau')\|_{-p_5}^{2j}] \leq C_{14}(T,h)\{|s-s'|^j + |t - t'|^j + |\tau-\tau'|^j\},$$
$$0 \leq s,s',t,t',\tau,\tau' \leq T.$$

First we will show the following inequality. Let $A(r)$ be a well measurable random linear operator from E_1' to E_{p_0}' such that

$E[\int_s^t \| A(r) \|_2^2 dr] < \infty$. Then we have

Lemma 3. For any integer $j \geq 1$,

$$E[\|\int_s^t A(r)d\beta(r)\|_{-P_0}^{2j}] \leq C_{15}(j)E[(\int_s^t \| A(r) \|_2^2 dr)^j].$$

Proof. Let $(\cdot,\cdot)_{-P_0}$ be the inner product in E'_{P_0} such that
$(x,x)_{-P_0} = \|x\|_{-P_0}^2$. Setting $\theta(x) = (x,x)_{-P_0}^j$ and $y(t) = \int_s^t A(r)d\beta(r)$ and applying
the Itô formula, (Kuo [15]), for $\theta(y(t))$, we get

$$(3.1) \qquad E[\|y(t)\|_{-P_0}^{2j}] = \frac{1}{2} E[\int_s^t \text{trace}_{E_0} D^2\theta(y(r))\circ[A(r)\times A(r)]dr]$$

$$= \frac{1}{2} E[\int_s^t \sum_{i=1}^{\infty} \{2^2 j(j-1)(y(r),A(r)h_i^{(0)})_{-P_0}^2 \|y(r)\|_{-P_0}^{2(j-2)}$$

$$+ 2j\|A(r)h_i^{(0)}\|_{-P_0}^2 \|y(r)\|_{-P_0}^{2(j-1)}\}dr]$$

$$\leq (j+2j(j-1))E[\int_s^t \| A(r) \|_2^2 \|y(r)\|_{-P_0}^{2(j-1)}dr].$$

By Hölder's inequality and the martingale inequality, the right hand side of
(3.1) is dominated by

$$(j+2j(j-1))E[\sup_{s\leq r\leq t} \|y(r)\|_{-P_0}^{2j}]^{j-1/j}E[(\int_s^t \| A(r) \|_2^2 dr)^j]^{1/j}$$

$$\leq (2j^2-j)(2j/(2j-1))^{2(j-1)}E[\|y(t)\|_{-P_0}^{2j}]^{j-1/j} E[(\int_s^t \| A(r) \|_2^2 dr)^j]^{1/j}.$$

which completes the proof of Lemma 3.

Proof of Lemma 2. Now for the convenience of notations we will write
$dt = d\beta_0(t)$, $d\beta(t) = d\beta_1(t)$, $A_0(t,x) = B(t,x)$, $A_1(t,x) = A(t,x)$, $\|\cdot\|_0 = \|\cdot\|_{-P_0}$
and $\|\cdot\|_1 = \|\cdot\|_2$. Without loss of generality, we may assume

$0 \leq s < s' < t < t' \leq T$. Then $\xi_{s,t}(\tau) - \xi_{s',t}(\tau')$ is a sum of the following terms:

$$(3.2) \qquad \sum_k \int_s^{s'} (\int_0^1 DA_k(r,\zeta_{s,r}(\tau,y))(\xi_{s,r}(\tau))dy)d\beta_k(r).$$

where $\zeta_{s,r}(\tau,y) = \eta_{s,r}(x) + y(\eta_{s,r}(x+\tau h) - \eta_{s,r}(x))$.

$$(3.3) \qquad \sum_k \int_s^t (\int_0^1 \{DA_k(r,\zeta_{s,r}(\tau,y))(\xi_{s,r}(\tau))$$

$$-DA_k(r,\zeta_{s',r}(\tau',y))(\xi_{s',r}(\tau'))\}dy)d\beta_k(r).$$

By Lemma 3 and the assumption (H2), the expectation of the $2j$-th power of the $\|\cdot\|_{-p_5}$-norm of (3.2) is dominated by

$$C_{16} \sum_k E[(\int_s^{s'} \|\int_0^1 DA_k(r,\zeta_{s,r}(\tau,y))(\xi_{s,r}(\tau))dy \|_k^2 dr)^j]$$

$$\leq C_{17} \sum_k |s'-s|^{j-1} E[\int_s^{s'} \|\xi_{s,r}(\tau)\|_{-p_5}^{2j} dr].$$

Again using Lemma 3, assumption (H2) and the Gronwall lemma, we have

$$(3.4) \qquad E[\|\eta_{s,t}(x)-\eta_{s,t}(y)\|_{-p_5}^{2j}] \leq C_{18}\|x-y\|_{-p_5}^{2j}, \quad x,y \in E'_{p_5}.$$

which implies

$$(3.5) \qquad E[\int_s^{s'} \|\xi_{s,r}(\tau)\|_{-p_5}^{2j} dr] \leq C_{18}\|h\|_{-p_5}^{2j} |s'-s|.$$

Since the integrand in (3.3)

$$= \int_0^1 DA_k(r,\zeta_{s,r}(\tau,y))(\xi_{s,r}(\tau) - \xi_{s',r}(\tau'))dy$$

$$+ \int_0^1 (\int_0^1 D^2 A_k(r,\tau_{s,s',r}(\tau,\tau',y_1))(\zeta_{s,r}(\tau,y) - \zeta_{s',r}(\tau',y))dy_1)(\xi_{s',r}(\tau'))dy.$$

where $\tau_{s,s',r}(\tau,\tau',y_1) = \zeta_{s',r}(\tau',y) + y_1(\zeta_{s,r}(\tau,y) - \zeta_{s',r}(\tau',y))$. the $\|\cdot\|_k$-norm of the integrand is dominated by

$$(3.6) \qquad C_{19} \{\|\xi_{s,r}(\tau) - \xi_{s',r}(\tau')\|_{-p_5} + (\|\eta_{s,r}(x) - \eta_{s',r}(x)\|_{-p_5}$$

$$+ \|\eta_{s,r}(x+\tau h) - \eta_{s',r}(x+\tau' h)\|_{-p_5}) \|\xi_{s',r}(\tau')\|_{-p_5}\}.$$

By Lemma 3 and (3.6), the expectation of the 2j-th power of $\|\cdot\|_{-p_5}$-norm of (3.3) is dominated by

$$(3.7) \quad C_{20}\{\int_s^t E[\|\xi_{s,r}(\tau) - \xi_{s',r}(\tau')\|_{-p_5}^{2j}]dr$$

$$+ \int_s^t E[\|\eta_{s,r}(x) - \eta_{s',r}(x)\|_{-p_5}^{4j}]^{1/2} E[\|\xi_{s',r}(\tau')\|_{-p_5}^{4j}]^{1/2} dr$$

$$+ \int_s^t E[\|\eta_{s,r}(x+\tau h) - \eta_{s',r}(x+\tau' h)\|_{-p_5}^{4j}]^{1/2} E[\|\xi_{s',r}(\tau')\|_{-p_5}^{4j}]^{1/2} dr\}.$$

From the assumptions (H1) and (H2), we get

$$\|A_k(r,\eta_{s,r}(x)) - A_k(r,\eta_{s',r}(x'))\|_k \le C_{21}\|\eta_{s,r}(x) - \eta_{s',r}(x')\|_{-p_5}$$

and taking the expectations of the 2n-th power of both sides of $\|\cdot\|_{-p_5}$-norm of the following inequality;

$$\|\eta_{s,t}(x) - \eta_{s',t'}(x')\|_{-p_5}$$

$$\le \|\sum_k \int_s^{s'} A_k(r,\eta_{s,r}(x))d\beta_k(r)\|_{-p_5}$$

$$+ \|\sum_k \int_t^{t'} A_k(r,\eta_{s',r}(x'))d\beta_k(r)\|_{-p_5}$$

$$+ \|\sum_k \int_s^t \{A_k(r,\eta_{s,r}(x)) - A_k(r,\eta_{s',r}(x'))\}d\beta_k(r)\|_{-p_5}.$$

we have, by Lemma 3, similarly

$$E[\|\eta_{s,t}(x) - \eta_{s',t'}(x')\|_{-P_5}^{2n}]$$

$$\leq C_{22}(T)\{|t-t'|^n + |s-s'|^n + \int_s^t E[\|\eta_{s,r}(x)-\eta_{s',r}(x')\|_{-P_5}^{2n}]dr\}.$$

Noticing that $\eta_{s,r}(x) = \eta_{s',r}(\eta_{s,s'}(x))$ and $\eta_{s',r}(\cdot)$ is independent of $\eta_{s,s'}(\cdot)$ and using (3.4), we get

$$E[\|\eta_{s,r}(x) - \eta_{s',r}(x')\|_{-P_5}^{2n}] = \int_{E_{P_5}'} E[\|\eta_{s',r}(y)-\eta_{s',r}(x')\|_{-P_5}^{2n}]P(\eta_{s,s'}(x)\in dy)$$

$$\leq \int_{E_{P_5}'} C_{23}\|y-x'\|_{-P_5}^{2n}P(\eta_{s,s'}(x) \in dy)$$

$$= C_{23}E[\|\eta_{s,s'}(x) - x'\|_{-P_5}^{2n}]$$

$$\leq C_{24}\{\|x-x'\|_{-P_5}^{2n} + |s'-s|^n\}.$$

where $P(\cdot)$ denotes the fundamental probability measure associated with $\beta(t)$. Hence we obtain

(3.8) $E[\|\eta_{s,t}(x) - \eta_{s',t'}(x')\|_{-P_5}^{2n} \leq C_{25}(T)\{|t-t'|^n+|s-s'|^n+\|x-x'\|_{-P_5}^{2n}\}.$

Combining (3.2), (3.3), (3.4), (3.5), (3.7) and (3.8), we have

$$E[\|\xi_{s,t}(\tau) - \xi_{s',t'}(\tau')\|_{-P_5}^{2j}]$$

$$\leq C_{26}(T)\|h\|_{-P_5}^{2j}\{|t-t'|^j + |s-s'|^j + |\tau-\tau'|^{2j}\|h\|_{-P_5}^{2j}\}.$$

This completes the proof of Lemma 2.

Let τ tend to 0, we have for each $x \in E_p'$.

(3.9) $D\eta_{s,t}(x)(h) = h + \sum_k \int_s^t DA_k(r,\eta_{s,r}(x))(D\eta_{s,r}(x)(h))d\beta_k(r).$

For the higher order differentiations, the formula similar to (3.9) can be proved inductively, together with the following lemma.

Lemma 4. Suppose that a natural number $q \geq p_0$ and any $T > 0$. Then for $0 \leq s, t, s', t' \leq T$, a natural number j and $x, x', h_i \in E_q'$, $i=1, 2, \cdots, n$, we have

(3.10) $\quad E[\|D^n \eta_{s,t}(x)(h_1, h_2, \cdots, h_n)\|_{-q}^{2j}] \leq C_{27}(T)\|h_1\|_{-q}^{2j}\|h_2\|_{-q}^{2j} \cdots \|h_n\|_{-q}^{2j}$.

(3.11) $\quad E[\|D^n \eta_{s,t}(x)(h_1, h_2, \cdots, h_n) - D^n \eta_{s',t'}(x')(h_1, h_2, \cdots, h_n)\|_{-q}^{2j}]$

$$\leq C_{28}(T)\{|t-t'|^j + |s-s'|^j + \|x-x'\|_{-q}^{2j}\} \|h_1\|_{-q}^{2j}\|h_2\|_{-q}^{2j} \cdots \|h_n\|_{-q}^{2j} .$$

Proof. First we will show (3.10) for the case $n=1$. By the assumptions (H1) and (H2), we get

$$\|DA_k(r, \eta_{s,r}(x))(D\eta_{s,r}(x)(h))\|_k \leq C_{29}\|D\eta_{s,r}(x)(h)\|_{-q}.$$

so that taking the expectations of $2j$-th powers of $\|\cdot\|_{-q}$ norms of both sides of (3.9) and using Lemma 3, we get

$$E[\|D\eta_{s,t}(x)(h)\|_{-q}^{2j}] \leq C_{30}(T)\{\|h\|_{-q}^{2j} + \int_s^t E[\|D\eta_{s,r}(x)(h)\|_{-q}^{2j}]dr\}$$

and the Gronwall inequality gives (3.10) for the case where $n=1$. For $n \geq 2$, we will prove the inequality by mathematical induction. For $h_1, h_2, \cdots, h_n \in E_q'$,

$$(D^n \eta_{s,t}(x))(h_1, h_2, \cdots, h_n) = \sum_k \int_s^t D^n(A_k(r, \eta_{s,r}(x)))(h_1, h_2, \cdots, h_n)d\beta_k(r).$$

Since

(3.12) $\quad D^n(A_k(r, \eta_{s,r}(x)))(h_1, h_2, \cdots, h_n)$

$$= DA_k(r, \eta_{s,r}(x))(D^n \eta_{s,r}(x)(h_1, h_2, \cdots, h_n))$$

$$+ \text{ finite sum of terms of the type}$$

$$(D^m A_k(r, \eta_{s,r}(x)))(D^{n_1} \eta_{s,r}(x)(h_{j_1^{(1)}}, h_{j_2^{(1)}}, \cdots, h_{j_{n_1}^{(1)}}).$$

$$D^{n_2} \eta_{s,r}(x)(h_{j_1^{(2)}}, h_{j_2^{(2)}}, \cdots, h_{j_{n_2}^{(2)}}), \cdots, D^{n_m} \eta_{s,r}(x)(h_{j_1^{(m)}}, h_{j_2^{(m)}}, \cdots, h_{j_{n_m}^{(m)}})).$$

where $2 \leq m \leq n$, $n_1 + n_2 + \cdots + n_m = n$ and $0 \leq n_1 \leq n-1$, so that using the inductive assumption, we get (3.10) by the same argument as before.

Before proceeding to the proof of (3.11), we note that for $h \in E_q'$, $\|D\eta_{s,t}(x)(h) - D\eta_{s',t'}(x')(h)\|_{-q}$ is dominated by

$$(3.13) \qquad \sum_k \|\int_s^{s'} D(A_k(r, \eta_{s,r}(x)))(h) d\beta_k(r)\|_{-q}$$

$$+ \sum_k \|\int_t^{t'} D(A_k(r, \eta_{s',r}(x')))(h) d\beta_k(r)\|_{-q}$$

$$+ \sum_k \|\int_s^t \{D(A_k(r, \eta_{s,r}(x)))(h) - D(A_k(r, \eta_{s',r}(x')))(h)\} d\beta_k(r)\|_{-q}.$$

Now, by the assumptions (H1) and (H2), we have

$$(3.14) \qquad \|D(A_k(r, \eta_{s,r}(x)))(h) - D(A_k(r, \eta_{s',r}(x')))(h)\|_k$$

$$\leq \|\{DA_k(r, \eta_{s,r}(x)) - DA_k(r, \eta_{s',r}(x'))\}(D\eta_{s,r}(x)(h))\|_k$$

$$+ \|DA_k(r, \eta_{s',r}(x'))(D\eta_{s,r}(x)(h) - D\eta_{s',r}(x')(h))\|_k$$

$$\leq C_{31}(T)\{\|\eta_{s,r}(x) - \eta_{s',r}(x')\|_{-q} \|D\eta_{s,r}(x)(h)\|_{-q}$$

$$+ \|D\eta_{s,r}(x)(h) - D\eta_{s',r}(x')(h)\|_{-q}\}.$$

Hence from (3.8), (3.13) and (3.14) we have

$$E[\|D\eta_{s,t}(x)(h) - D\eta_{s',t'}(x')(h)\|_{-q}^{2j}]$$

$$\leq C_{32}(T)\{((|t-t'|^J + |s-s'|^J + \|x-x'\|_{-q}^{2J})\|h\|_{-q}^{2J}$$

$$+ \int_s^t E[\|D\eta_{s,t}(x)(h) - D\eta_{s',r}(x')(h)\|_{-q}^{2J}]dr\}.$$

which gives (3.11) by the Gronwall lemma for the case n=1. By (3.12) and the estimation of $\|D^n\eta_{s,t}(x)(h_1,h_2,\cdots,h_n) - D^n\eta_{s',t'}(x)(h_1,h_2,\cdots,h_n)\|_{-q}$ similar to that in (3.13), mathematical induction and the Gronwall lemma yield the proof of (3.11) for $n \geq 2$.

For the proof of the generation problem of $L(t)$ we proceed as follows. By the assumptions (H1) and (H2), (3.8) and (3.10) of Lemma 4, we may exchange the order of the differentiation and the integration. Then by the Itô formula [15], we have the pointwise Kolmogorov forward and backward equations as in the finite dimensional case (Theorem 1 (page 73) of [7]):

$$\frac{d}{dt}(U(t,s)F)(x) = (U(t,s)L(t)F)(x)$$

$$\frac{d}{ds}(U(t,s)F)(x) = -(L(s)U(t,s)F)(x).$$

Let $p \geq 0$, $q \geq 0$ and $n \geq 0$ be integers and $x \in E_p'$. Since $D^n(F(\eta_{s,t}(x)))(h_{i_1}^{(q)}, h_{i_2}^{(q)}, \cdots, h_{i_n}^{(q)})$ is a finite sum of terms of the type

$$I = D^m F(\eta_{s,t}(x))(D^{n_1}\eta_{s,t}(x)(h_{j_1^{(1)}}^{(q)}, h_{j_2^{(1)}}^{(q)}, \cdots, h_{j_{n_1}^{(1)}}^{(q)}), D^{n_2}\eta_{s,t}(x)$$

$$(h_{j_1^{(2)}}^{(q)}, h_{j_2^{(2)}}^{(q)}, \cdots, h_{j_{n_2}^{(2)}}^{(q)}), \cdots, D^{n_m}\eta_{s,t}(x)(h_{j_1^{(m)}}^{(q)}, h_{j_2^{(m)}}^{(q)}, \cdots, h_{j_{n_m}^{(m)}}^{(q)})).$$

$$n_1 + n_2 + \cdots + n_m = n,$$

so that from the nuclearity of E and (3.10), we have an integer $q' > \max\{p, p_0, q\}$ such that

$$(3.15) \qquad \sum_{j=1}^{\infty} \|h_j^{(q)}\|_{-q'}^2 < +\infty$$

and

$$(3.16) \quad E[|I|^2] \le \|F\|_{q',q',n}^2 E[e^{2\|\eta_{s,t}(x)\|_{-q'}} \|D^{n_1}\eta_{s,t}(x)(h_{j_1^{(1)}}^{(q)}, h_{j_2^{(1)}},$$

$$\cdots, h_{j_{n_1}^{(1)}}^{(q)})\|_{-q}^2 \cdot \|D^{n_2}\eta_{s,t}(x)(h_{j_1^{(2)}}^{(q)}, h_{j_2^{(2)}}^{(q)}, \cdots, h_{j_{n_2}^{(2)}}^{(q)})\|_{-q}^2 \cdots$$

$$\cdots \|D^{n_m}\eta_{s,t}(x)(h_{j_1^{(m)}}^{(q)}, h_{j_2^{(m)}}^{(q)}, \cdots, h_{j_{n_m}^{(m)}}^{(q)})\|_{-q}^2]$$

$$\le C_{33}\|F\|_{q',q',n}^2 \|h_{i_1}^{(q)}\|_{-q}^2 \cdot \|h_{i_2}^{(q)}\|_{-q}^2 \cdots \|h_{i_n}^{(q)}\|_{-q}^2 \cdot E[e^{4\|\eta_{s,t}(x)\|_{-q'}}]^{\frac{1}{2}}.$$

Here we will prove

Lemma 5. For any $\alpha > 0$ and $T > 0$, there exists a constant $C_{34} = C_{34}(\alpha, T)$ such that

$$\sup_{0 \le s, t \le T} E[e^{\alpha\|\eta_{s,t}(x)\|_{-q'}}] \le C_{34}e^{\alpha\|x\|_{-q'}}.$$

Proof. By (H1), $\|\eta_{s,t}(x)\|_{-q'} \le \|x\|_{-q'} + C_{35} + \|\int_s^t A(r, \eta_{s,r}(x))d\beta(r)\|_{-q'}$. Following [8], it is enough to prove $E[\exp(\|\int_s^t \alpha A(r, \eta_{s,r}(x))d\beta(r)\|_{-q'})] \le C_{36}$. Setting $y_{s,t}(x) = \int_s^t \alpha A(r, \eta_{s,r}(x))d\beta(r)$, by the Itô formula and the assumption (H1), we get for any integer $m \ge 2$,

$$(3.17) \qquad E[\|y_{s,t}(x)\|_{-q}^m] \le E[(1+\|y_{s,t}(x)\|_{-q}^2)^{m/2}]$$

$$\le E[1 + \int_s^t \frac{1}{2}(2\frac{m}{2}(1+\|y_{s,r}(x)\|_{-q}^2)^{\frac{m}{2}-1}\alpha^2\|A(r, \eta_{s,r}(x))\|_2^2$$

$$+ 4\frac{m}{2}(\frac{m}{2}-1)(1+\|y_{s,r}(x)\|^2_{-q'})^{\frac{m}{2}-2} \alpha^2 (\sum_{i=1}^{\infty} (y_{s,r}(x),A(r,\eta_{s,r}(x))h_i^{(0)})^2_{-q'})\}dr]$$

$$\leq 1 + 2(\frac{m}{2})^2\alpha^2 C_{37}\int_s^t E[(1+\|y_{s,r}(s)\|^2_{-q'})^{\frac{m}{2}-1}]dr.$$

where $C_{37} = \sup\limits_{\substack{x\in E' \\ 0\leq t\leq T}} \| A(t,x) \|^2_2$ and $\|x\|^2_{-q'} = (x,x)_{-q'}$. If we use (3.17)

recursively, the rest is similar to the argument in [8], which completes the

proof.

Therefore (3.15), (3.16) and Lemma 5 yield

$$\| U(t,s)F\|_{p,q,n} \leq C_{38}(T)\|F\|_{q',q',n} \quad , \quad t,s \in [0,T],$$

which implies that $U(t,s)$ is a continuous linear operator from $\mathfrak{D}_E$ into itself.

In the same way as in [21], if we prove the strong continuity of $U(t,s)F$

in (t,s), the pointwise Kolmogorov forward and backward equations imply that

$L(t)$ generates the evolution operator $U(t,s)$. Since $\|U(t,s)^{\top} U(t',s')F\|^{2j}_{p,q,n}$

is dominated by a finite sum of terms of the type

$$\sup_{x\in E'_p} e^{-2j\|x\|_{-p}} \sum_{\substack{j_1^{(1)},j_2^{(1)},\ldots,j_{m_1}^{(1)} \\ \vdots \\ j_1^{(m)},j_2^{(m)},\ldots,j_{n_m}^{(m)}}} E[|D^m F(\eta_{s,t}(x))(D^{n_1}\eta_{s,t}(x)$$

$$(h^{(q)}_{j_1^{(1)}},h^{(q)}_{j_2^{(1)}},\ldots,h^{(q)}_{j_{n_1}^{(1)}}), D^{n_2}\eta_{s,t}(x)(h^{(q)}_{j_1^{(2)}},h^{(q)}_{j_2^{(2)}},\ldots,h^{(q)}_{j_{n_2}^{(2)}}),\ldots.$$

$$D^{n_m}\eta_{s,t}(x)(h^{(q)}_{j_1^{(m)}}, h^{(q)}_{j_2^{(m)}},\ldots,h^{(q)}_{j_{n_m}^{(m)}})) - D^m F(\eta_{s',t'}(x))(D^{n_1}\eta_{s',t'}(x)$$

$$(h^{(q)}_{j^{(1)}_1}, h^{(q)}_{j^{(1)}_2}, \cdots, h^{(q)}_{j^{(1)}_{n_1}}), D^{n_2}\eta_{s',t'}(x)(h^{(q)}_{j^{(2)}_1}, h^{(q)}_{j^{(2)}_2}, \cdots, h^{(q)}_{j^{(2)}_{n_2}}),$$

$$\cdots, D^{n_m}\eta_{s',t'}(x)(h^{(q)}_{j^{(m)}_1}, h^{(q)}_{j^{(m)}_2}, \cdots, h^{(q)}_{j^{(m)}_{n_m}}))|^{2J}].$$

so that by (3.8), Lemmas 4 and 5 and the nuclearity of E, we have

$$\|U(t,s)F - U(t',s')F\|^{2J}_{p,q,n} \le C_{39}\|F\|^{2J}_{q',q',n+1}\{|t-t'|^J + |s-s'|^J\}.$$

This completes the proof of Lemma 1.

§4. Generation of the Kolmogorov Evolution Operator

In this Section, we will discuss assumption (H4). Let K be a separable Hilbert space. We call a K-valued functional $G(x) = g(\langle x,\xi_1\rangle, \langle x,\xi_2\rangle, \cdots, \langle x,\xi_n\rangle)$, $\xi_1, \xi_2, \cdots, \xi_n \in E$, a smooth functional if $g(x): \mathbb{R}^n \to K$ is a C^∞-function. Further we call $G(x)$ a bounded smooth functional if $g(x)$ itself and all the derivatives of $g(x)$ are bounded. The coefficients $A(t,x)$ and $B(t,x)$ are said to be approximated by bounded smooth functionals on E' if for any integers, $p \ge p_0$, $q \ge 0$ and $n \ge 0$, there exist sequences of bounded smooth functionals

$$A_m(t,x) = a_m(t, \langle x,\xi_1\rangle, \langle x,\xi_2\rangle, \cdots, \langle x,\xi_{k_m}\rangle)$$

and

$$B_m(t,x) = b_m(t, \langle x,\xi_1\rangle, \langle x,\xi_2\rangle, \cdots, \langle x,\xi_{k_m}\rangle)$$

such that the following conditions are satisfied:

(4.1) $A_m(t,x)$ and $B_m(t,x)$ satisfy the conditions (H_1), (H_2) and (H_3).

(4.2)   For any $T > 0$,

$$\lim_{\substack{m\to\infty \\ \substack{x\in E'_p \\ 0\leq t\leq T}}} \sup \| A(t,x)-A_m(t,x) \|_2^2 = 0,$$

$$\lim_{\substack{m\to\infty \\ \substack{x\in E'_p \\ 0\leq t\leq T}}} \sup \| B(t,x)-B_m(t,x) \|_{-p_0} = 0,$$

$$\lim_{\substack{m\to\infty \\ \substack{x\in E'_p \\ 0\leq t\leq T}}} \sup \| D^k A(t,x)-D^k A_m(t,x) \|_{H.S.}^{(q)} = 0, \quad k=1,2,\cdots,n,$$

$$\lim_{\substack{m\to\infty \\ \substack{x\in E'_p \\ 0\leq t\leq T}}} \sup \| D^k B(t,x)-D^k B_m(t,x) \|_{H.S.}^{(q)} = 0, \quad k=1,2,\cdots,n.$$

Proposition 2. Suppose that the coefficients $A(t,x)$ and $B(t,x)$ are approximated by bounded smooth functionals on E'. Then if $F \in \mathcal{D}_{E'}$,

$$U(t,s)F(x) = E[F(\eta_{s,t}(x))] \in \mathcal{D}_{E'}.$$

Proof. It is convenient to use the notation $A_0(t,x) = B(t,x)$ and $A_1(t,x) = A(t,x)$. For any integers $p \geq 0$, $q \geq 0$ and $n \geq 0$, we choose an integer $q' > \max\{p,p_0,q\}$ such that

$$(4.3) \qquad \sum_{j=1}^{\infty} \| h_j^{(q)} \|_{-q'}^2 < +\infty,$$

since E is a nuclear Fréchet space. Then by the assumptions, for any $0 < \delta < 1$ and $A_k(t,x)$, $k=0,1$, there exist bounded smooth functionals $\tilde{A}_k(t,x) = \tilde{a}_k(t,\langle x,\zeta_1\rangle,\langle x,\zeta_2\rangle,\cdots,\langle x,\zeta_{m_k}\rangle)$, $k=0,1$ such that

$$(4.4) \qquad \sum_{\substack{\ell=0 \\ \substack{x\in E'_{q'} \\ 0\leq t\leq T}}}^{n+1} \sup \| D^\ell A_k(t,x)-D^\ell \tilde{A}_k(t,x) \|_{H.S.}^{(q')} < \delta.$$

For sufficiently large N, we put

$$z_{s,t}^N(x) = x + \sum_k \int_s^t \widetilde{A}_k(t_1, x + \sum_k \int_s^{t_1} \widetilde{A}_k(t_2, \cdots$$

$$\cdots, x + \sum_k \int_s^{t_{N-1}} \widetilde{A}_k(t_N, x) d\beta_k(t_N)) \cdots) d\beta_k(t_1).$$

Setting

$$\hat{z}_{s,t}^{(n)}(x) = x + \sum_k \int_s^t \widetilde{A}_k(t_1, x + \sum_k \int_s^{t_1} \widetilde{A}_k(t_2, \cdots, x + \sum_k \int_s^{t_{n-1}} \widetilde{A}_k(t_n, \eta_{s,t_n}(x)) d\beta_k(t_n)) \cdots) d\beta_k(t_1).$$

$n = 1, 2, \cdots, N$, where $t_0 = t$, by Lemma 3, we have for any $x \in E_p'$, $0 \leq s, t \leq T$ and any integer $j \geq 1$,

$$(4.5) \qquad E[\|\eta_{s,t}(x) - z_{s,t}^N(x)\|_{-q'}^{2j}]$$

$$\leq 2^{2j-1} E[\|\eta_{s,t}(x) - \hat{z}_{s,t}^{(1)}(x)\|_{-q'}^{2j}]$$

$$+ \sum_{k=2}^N (2^{2j-1})^k E[\|\hat{z}_{s,t}^{(k-1)}(x) - \hat{z}_{s,t}^{(k)}(x)\|_{-q'}^{2j}]$$

$$+ (2^{2j-1})^N E[\|\hat{z}_{s,t}^{(N)}(x) - z_{s,t}^N(x)\|_{-q'}^{2j}]$$

$$\leq (2^{2j-1})^2 \delta^{2j} RT + \sum_{k=2}^N (2^{2j-1})^{2k} M^{2j(k-1)} \delta^{2j} R^k T^k / k!$$

$$+ (2^{2j-1})^{2N} R^N 2^{2j} M^{2jN} T^N / N!$$

$$\leq \delta^{2j} \exp(2^{2(2j-1)} R(MV1)^{2j} T) + (2^{2j-1})^{2N} R^N 2^{2j} M^{2jN} T^N / N!.$$

where $M = \max_k \max_{0 \leq \ell \leq n+1} \sup_{\substack{x \in E_q' \\ 0 \leq t \leq T}} \|D^\ell \widetilde{A}_k(t,x)\|_{H.S.}^{(q')}$ and $R = C_{15}(j) T^{j-1} + T^{2j-1}$. Hence

for any $\epsilon > 0$, if we take sufficiently small δ and large N, we have

$$(4.6) \qquad \sup_{x \in E_p'} E[\|\eta_{s,t}(x) - z_{s,t}^N(x)\|_{-q'}^{2j}] < \epsilon.$$

Next we will verify by mathematical induction that for any integer

$1 \leq k \leq n$ and any $\epsilon > 0$, there exists an integer $N(k,\epsilon)$ such that if $N \geq N(k,\epsilon)$,

$$(4.7) \qquad E[\|D^k \eta_{s,t}(x)(h_{i_1}^{(q)}, h_{i_2}^{(q)}, \cdots, h_{i_k}^{(q)})$$
$$- D^k z_{s,t}^N(x)(h_{i_1}^{(q)}, h_{i_2}^{(q)}, \cdots, h_{i_k}^{(q)})\|_{-q}^{2j}] < \epsilon.$$

For any $\epsilon' > 0$, (4.5) gives that if we take sufficiently small δ and large $n(\epsilon')$, then for any $N \geq n(\epsilon')$

$$(4.8) \qquad \sup_{x \in E_p'} E[\|\eta_{s,t}(x) - z_{s,t}^N(x)\|_{-q}^{4j}]^{1/2} < \epsilon'.$$

Here we need the following lemma for later use. In a manner similar to that in the proofs of (3.10) and Lemma 5, we get

Lemma 6. For any integers $q \geq p_0$, $j \geq 1$, $n \geq 1$ and any $T > 0$, we have

$$(4.9) \qquad \sup_{0 \leq s,t \leq T} E[\|D^n z_{s,t}^N(x)(h_1, h_2, \cdots, h_n)\|_{-q}^{2j}]$$
$$\leq C_{40}(T)\|h_1\|_{-q}^{2j}\|h_2\|_{-q}^{2j} \cdots \|h_n\|_{-q}^{2j}, \quad x, h_i, i = 1, 2, \cdots, n \in E_q'.$$

For any $\alpha > 0$ and $T > 0$,

$$(4.10) \qquad \sup_{0 \leq s,t \leq T} E[e^{\alpha\|z_{s,t}^N(x)\|_{-q}}] \leq C_{41} e^{\alpha\|x\|_{-q}}.$$

For any $\xi \in E$ and any $\alpha > 0$ and $T > 0$, there exists $C_{42} = C_{42}(\xi, \alpha, T)$ such that

$$(4.11) \qquad \sup_{0 \leq s,t \leq T} \max\{E[\exp(\alpha\sqrt{|\langle \eta_{s,t}(x), \xi \rangle|})], E[\exp(\alpha\sqrt{|\langle z_{s,t}^N(x), \xi \rangle|})]\}$$

$$\leq C_{42} \exp(\alpha\sqrt{|\langle x, \xi \rangle|}).$$

Setting

$$y_{s,t}^{m,N}(x)(h_{i_1}^{(q)})$$

$$= h_{i_1}^{(q)} + \sum_k \int_s^t \widetilde{DA}_k(t_1, z_{s,t_1}^{N-1}(x))(h_{i_1}^{(q)} + \sum_k \int_s^{t_1} \widetilde{DA}_k(t_2, z_{s,t_2}^{N-2}(x))(h_{i_1}^{(q)}$$

$$+\cdots+\sum_k \int_s^{t_{m-1}} \widetilde{DA}_k(t_m, \eta_{s,t_m}(x))(D\eta_{s,t_m}(x)(h_{i_1}^{(q)}))d\beta_k(t_m)\cdots)d\beta_k(t_1).$$

$$z_{s,t}^{m,N}(x)(h_{i_1}^{(q)})$$

$$= h_{i_1}^{(q)} + \sum_k \int_s^t \widetilde{DA}_k(t_1, z_{s,t_1}^{N-1}(x))(h_{i_1}^{(q)} + \sum_k \int_s^{t_1} \widetilde{DA}_k(t_2, z_{s,t_2}^{N-2}(x))(h_{i_1}^{(q)}$$

$$+\cdots+\sum_k \int_s^{t_{m-1}} \widetilde{DA}_k(t_m, z_{s,t_m}^{N-m}(x))(D\eta_{s,t_m}(x)(h_{i_1}^{(q)}))d\beta_k(t_m)\cdots)d\beta_k(t_1)$$

and taking $N \geq m + n(\epsilon')$, we have by Lemma 3, (3.10) and (4.9),

$$E[\|D\eta_{s,t}(x)(h_{i_1}^{(q)})-Dz_{s,t}^N(x)(h_{i_1}^{(q)})\|_{-q'}^{2J}]$$

$$\leq 2^{2J-1}E[\|D\eta_{s,t}(x)(h_{i_1}^{(q)})-y_{s,t}^{1,N}(x)(h_{i_1}^{(q)})\|_{-q'}^{2J}]$$

$$+ (2^{2J-1})^2 E[\|y_{s,t}^{1,N}(x)(h_{i_1}^{(q)})-z_{s,t}^{1,N}(x)(h_{i_1}^{(q)})\|_{-q'}^{2J}]$$

$$+ \sum_{k=1}^{m-1} \{(2^{2J-1})^{k+2}E[\|z_{s,t}^{k,N}(x)(h_{i_1}^{(q)})-y_{s,t}^{k+1,N}(x)(h_{i_1}^{(q)})\|_{-q'}^{2J}]$$

$$+ (2^{2J-1})^{k+3}E[\|y_{s,t}^{k+1,N}(x)(h_{i_1}^{(q)})-z_{s,t}^{k+1,N}(x)(h_{i_1}^{(q)})\|_{-q'}^{2J}]\}$$

$$+ (2^{2J-1})^{m+2}E[\|z_{s,t}^{m,N}(x)(h_{i_1}^{(q)})-Dz_{s,t}^N(x)(h_{i_1}^{(q)})\|_{-q'}^{2J}]$$

$$\leq C_{43}(h_{i_1}^{(q)})\{\delta^{2J}(2^{2J-1})^2 R_T + \epsilon'(2^{2J-1})^3 M^{2J} R_T$$

$$+ \sum_{k=1}^{m-1} ((2^{2J-1})^{2k+3}\delta^{2J}R^{k+1}M^{2Jk}$$

$$+ (2^{2j-1})^{2k+4} \epsilon' R^{k+1} M^{2j(k+1)})T^{k+1}/(k+1)!$$

$$+ (2^{2j-1})^{2m+2} R^m M^{2jm} T^m/m!\}$$

$$\leq C_{44}(\delta^{2j} + \epsilon' + (2^{2j-1})^{2m+2} R^m M^{2jm} T^m/m!),$$

which gives (4.7) for k=1 if we take sufficiently small δ, ϵ' and large m. We assume (4.7) holds for integers $1 \leq k \leq \ell$, $\ell \geq 1$.

Since

$$D^{\ell+1}(A_k(r,\eta_{s,r}(x)))(h_{i_1}^{(q)}, h_{i_2}^{(q)}, \cdots, h_{i_{\ell+1}}^{(q)})$$

$$= DA_k(r,\eta_{s,r}(x))(D^{\ell+1}\eta_{s,r}(x)(h_{i_1}^{(q)}, h_{i_2}^{(q)}, \cdots, h_{i_{\ell+1}}^{(q)}))$$

+ finite sum of terms of the type

$$D^u A_k(r,\eta_{s,r}(x))(D^{n_1}\eta_{s,r}(x)(h_{j_1^{(1)}}^{(q)}, h_{j_2^{(1)}}^{(q)}, \cdots, h_{j_{n_1}^{(1)}}^{(q)}).$$

$$D^{n_2}\eta_{s,r}(x)(h_{j_1^{(2)}}^{(q)}, h_{j_2^{(2)}}^{(q)}, \cdots, h_{j_{n_2}^{(2)}}^{(q)}), \cdots.$$

$$D^{n_u}\eta_{s,r}(x)(h_{j_1^{(u)}}^{(q)}, h_{j_2^{(u)}}^{(q)}, \cdots, h_{j_{n_u}^{(u)}}^{(q)})).$$

where

$$2 \leq u \leq \ell+1, \quad n_1+n_2+\cdots+n_u = \ell+1, \quad \{h_{j_{n_i}^{(i)}}^{(q)}, \ i=1,2,\cdots,u\} = \{h_{i_j}^{(q)}, \ j=1,2,\cdots,\ell+1\}$$

and

$$D^{\ell+1}\eta_{s,t}(x)(h_{i_1}^{(q)}, h_{i_2}^{(q)}, \cdots, h_{i_{\ell+1}}^{(q)})$$

$$= \sum_k \int_s^t D^{\ell+1}(A_k(r, \eta_{s,r}(x)))(h_{i_1}^{(q)}, h_{i_2}^{(q)}, \cdots, h_{i_{\ell+1}}^{(q)}) d\beta_k(r).$$

so (4.7) for $k \geq 2$ can be proved similarly.

Since $F \in \mathfrak{D}_{E'}$, for any $0 < \epsilon' < 1$, we have a weighted Schwartz functional $\tilde{F}(x) = f(\langle x, \xi_1 \rangle, \langle x, \xi_2 \rangle, \cdots, \langle x, \xi_m \rangle)$ such that

$$(4.12) \qquad \sum_{k=0}^{n+1} \sup_{x \in E_q'} e^{-\|x\|_{-q'}} \|D^k(F(x) - \tilde{F}(x))\|_{H.S.}^{(q')} < \epsilon'.$$

Then to prove Proposition 2, it is enough to show $(U(t,s)F)(x)$ is approximated by weighted Schwartz functionals in $\|\cdot\|_{p,k}^{(q)}$, $0 \leq k \leq n$. Since $D^k(F(\eta_{s,t}(x)))(h_{i_1}^{(q)}, h_{i_2}^{(q)}, \cdots, h_{i_k}^{(q)})$ is a finite sum of terms of the type

$$(4.13) \qquad I_{h_{i_1}^{(q)}, h_{i_2}^{(q)}, \cdots, h_{i_k}^{(q)}}(\eta_{s,t}(x))$$

$$= D^u F(\eta_{s,t}(x))(D^{n_1}\eta_{s,t}(x)(h_{j_1^{(1)}}^{(q)}, h_{j_2^{(1)}}^{(q)}, \cdots, h_{j_{n_1}^{(1)}}^{(q)}), D^{n_2}\eta_{s,t}(x)$$

$$(h_{j_1^{(2)}}^{(q)}, h_{j_2^{(2)}}^{(q)}, \cdots, h_{j_{n_2}^{(2)}}^{(q)}), \cdots, D^{n_u}\eta_{s,t}(x)(h_{j_1^{(u)}}^{(q)}, h_{j_2^{(u)}}^{(q)}, \cdots, h_{j_{n_u}^{(u)}}^{(q)})).$$

where $0 \leq u \leq k$ and $n_1 + n_2 + \cdots + n_u = k$, so that setting

$$(4.14) \qquad J_{h_{i_1}^{(q)}, h_{i_2}^{(q)}, \cdots, h_{i_k}^{(q)}}(z_{s,t}^N(x))$$

$$= D^u \tilde{F}(z_{s,t}^N(x))(D^{n_1}z_{s,t}^N(x)(h_{j_1^{(1)}}^{(q)}, h_{j_2^{(1)}}^{(q)}, \cdots, h_{j_{n_1}^{(1)}}^{(q)}), D^{n_2}z_{s,t}^N(x)$$

$$(h_{j_1^{(2)}}^{(q)}, h_{j_2^{(2)}}^{(q)}, \cdots, h_{j_{n_2}^{(2)}}^{(q)}), \cdots, D^{n_u}z_{s,t}^N(x)(h_{j_1^{(u)}}^{(q)}, h_{j_2^{(u)}}^{(q)}, \cdots, h_{j_{n_u}^{(u)}}^{(q)}).$$

we see that $(\|U(t,s)F - E[\tilde{F}(z_{s,t}^N(\cdot))]\|_{p,k}^{(q)})^2$ is dominated by a finite sum of terms of the type

$$(4.15) \quad C_{45} \sup_{x \in E_p'} e^{-2\|x\|_{-p}} \sum_{i_1, i_2, \cdots, i_k = 1}^{\infty} E[|I_{h_{i_1}^{(q)}, h_{i_2}^{(q)}, \cdots, h_{i_k}^{(q)}}(\eta_{s,t}(x))$$

$$- J_{h_{i_1}^{(q)}, h_{i_2}^{(q)}, \cdots, h_{i_k}^{(q)}}(z_{s,t}^N(x))|^2]$$

$$\leq C_{46}\{\sup_{x \in E_p'} e^{-2\|x\|_{-p}} \sum_{i_1, i_2, \cdots, i_k = 1}^{\infty} E[e^{2\|\eta_{s,t}(x)\|_{-q'}}(\epsilon')^2 \|D^{n_1}\eta_{s,t}(x)(h_{j_1^{(1)}}^{(q)}, $$

$$h_{j_2^{(1)}}^q, \cdots, h_{j_{n_1}^{(1)}}^{(q)}\|_{-q'}^2 \cdot \|D^{n_2}\eta_{s,t}(x)(h_{j_1^{(2)}}^{(q)}, h_{j_2^{(2)}}^{(q)}, \cdots, h_{j_{n_2}^{(2)}}^{(q)})\|_{-q'}^2 \cdot$$

$$\cdots \|D^{n_u}\eta_{s,t}(x)(h_{j_1^{(u)}}^{(q)}, h_{j_2^{(u)}}^{(q)}, \cdots, h_{j_{n_u}^{(u)}}^{(q)})\|_{-q'}^2]$$

$$+ \sup_{x \in E_p'} e^{-2\|x\|_{-p}} \sum_{i_1, i_2, \cdots, i_k = 1}^{\infty} E[|J_{h_{i_1}^{(q)}, h_{i_2}^{(q)}, \cdots, h_{i_k}^{(q)}}(\eta_{s,t}(x))$$

$$- J_{h_{i_1}^{(q)}, h_{i_2}^{(q)}, \cdots, h_{i_k}^{(q)}}(z_{s,t}^N(x))|^2]\}.$$

Lemmas 5 and 6 and (4.12) give

$$(4.16) \quad \sup_{x \in E_p'} e^{-\|x\|_{-p}} \max\{E[(\|D^u\tilde{F}(z_{s,t}^N(x))\|_{H.S.}^{(q')})^2]^{1/2}, E[(\|D^{u+1}\tilde{F}(z_{s,t}^N(x)$$

$$+ \tau(\eta_{s,t}(x) - z_{s,t}^N(x)))\|_{H.S.}^{(q')})^2]^{1/2}\} \leq C_{47}(T), \ 0 \leq \tau \leq 1, \ 0 \leq s,t \leq T.$$

Hence from (3.10), (4.3), (4.16) and Lemma 5, we have constants C_{48} and C_{49} independent of ϵ', and for any $\epsilon > 0$, a natural number N_0 such that (4.15) is

dominated by

$$(4.17) \quad \epsilon/3 + C_{48}\epsilon' + C_{49} \sum_{i_1,i_2,\cdots,i_k=1}^{N_0} E[\|\eta_{s,t}(x)-z_{s,t}^N(x)\|_{-q}^2 \cdot \|D^{n_1}\eta_{s,t}(x)(h_{j_1^{(1)}}^{(q)},$$

$$h_{j_2^{(1)}}^{(q)},\cdots,h_{j_{n_1}^{(1)}}^{(q)})\|_{-q}^2 \cdots \|D^{n_u}\eta_{s,t}(x)(h_{j_1^{(u)}}^{(q)}, h_{j_2^{(u)}}^{(q)},\cdots,h_{j_{n_u}^{(u)}}^{(q)})\|_{-q}^2 \cdot$$

$$+ \sum_{r=1}^{u} \|D^{n_1}z_{s,t}^N(x)(h_{j_1^{(1)}}^{(q)},h_{j_2^{(1)}}^{(q)},\cdots,h_{j_{n_1}^{(1)}}^{(q)})\|_{-q}^2 \cdots \|D^{n_{r-1}}z_{s,t}^N(x)(h_{j_1^{(r-1)}}^{(q)},$$

$$h_{j_2^{(r-1)}}^{(q)},\cdots,h_{j_{n_{r-1}}^{(r-1)}}^{(q)})\|_{-q}^2 \cdot \|D^{n_r}\eta_{s,t}(x)(h_{j_1^{(r)}}^{(q)},h_{j_2^{(r)}}^{(q)},\cdots,h_{j_{n_r}^{(r)}}^{(q)})$$

$$- D^{n_r}z_{s,t}^N(x)(h_{j_1^{(r)}}^{(q)},h_{j_2^{(r)}}^{(q)},\cdots,h_{j_{n_r}^{(r)}}^{(q)})\|_{-q}^2 \cdot \|D^{n_{r+1}}\eta_{s,t}(x)(h_{j_1^{(r+1)}}^{(q)},$$

$$h_{j_2^{(r+1)}}^{(q)},\cdots,h_{j_{n_{r+1}}^{(r+1)}}^{(q)})\|_{-q}^2 \cdots \|D^{n_u}\eta_{s,t}(x)(h_{j_1^{(u)}}^{(q)},h_{j_2^{(u)}}^{(q)},\cdots,h_{j_{n_u}^{(u)}}^{(q)})\|_{-q}^2 \cdot].$$

Therefore noting (3.10), (4.6), (4.7), (4.9), (4.15), and (4.17) and taking sufficiently small ϵ', δ and large N, we obtain

$$\sup_{x\in E_p'} e^{-\|x\|} {}^{-p}\|D^k((U(t,s)F)(x))-D^k(E[\tilde{F}(z_{s,t}^N(x))])\|_{H.S.}^{(q)} < \epsilon.$$

The rest is to prove that $E[\tilde{F}(z_{s,t}^N(x))]$ is a weighted Schwartz functional. Of course $E[\tilde{F}(z_{s,t}^N(x))] = \phi_{s,t}(\langle x,\xi_1\rangle,\langle x,\xi_2\rangle,\cdots,\langle x,\xi_m\rangle,\langle x,\zeta_1\rangle,\langle x,\zeta_2\rangle,\cdots,\langle x,\zeta_\ell\rangle)$ is a smooth functional. To prove $g(x)\phi_{s,t}(x) \in \mathcal{Y}(\mathbb{R}^{\ell+m})$, by the Leibniz formula, it is sufficient to examine the finiteness of

$$\sup_{x\in\mathbb{R}^{\ell+m}} (1+|x|^2)^n |(\tfrac{d}{dx})^r g(x)(\tfrac{d}{dx})^k \phi_{s,t}(x)|, \text{ for any integers } 0\leq r,k\leq n.$$

By the expression (4.14) of $D^k(\tilde{F}(z_{s,t}^N(x)))(h_{i_1}^{(q)}, h_{i_2}^{(q)}, \cdots, h_{i_k}^{(q)})$, (4.9) and the

fact that $f(x) = h(x)\varphi(x)$, $x \in \mathbb{R}^m$ and $|(\frac{d}{dx})^r g(x)| \leq C_{50}\exp(-\sum_{i=1}^{\ell+m}\sqrt{|x_i|})$, it is

enough to show the finiteness of

$$(4.18) \quad \sup_Q (1 + \sum_{i=1}^m \langle x,\xi_i\rangle^2 + \sum_{j=1}^\ell \langle x,\zeta_j\rangle^2)^n \exp(-\sum_{i=1}^\ell \sqrt{|\langle x,\xi_i\rangle|} - \sum_{j=1}^m \sqrt{|\langle x,\zeta_j\rangle|})$$

$$\times E[(\overline{h}^{(\mu)}(z_{s,t}^N(x))\overline{\varphi}^{(v)}(z_{s,t}^N(x)))^2]^{1/2},$$

where

$$Q = \{x; (\langle x,\xi_1\rangle, \langle x,\xi_2\rangle, \cdots, \langle x,\xi_m\rangle, \langle x,\zeta_1\rangle, \langle x,\zeta_2\rangle, \cdots, \langle x,\zeta_\ell\rangle) \in \mathbb{R}^{\ell+m}\},$$

$$h^{(\mu)}(x) = (\frac{d}{dx})^\mu h(x), \quad \varphi^{(v)}(x) = (\frac{d}{dx})^v \varphi(x), \quad x \in \mathbb{R}^m,$$

$$\overline{h}^{(\mu)}(z_{s,t}^N(x)) = h^{(\mu)}(\langle z_{s,t}^N(x),\xi_1\rangle, \langle z_{s,t}^N(x),\xi_2\rangle, \cdots, \langle z_{s,t}^N(x),\xi_m\rangle)$$

and

$$\overline{\varphi}^{(v)}(z_{s,t}^N(x)) = \varphi^{(v)}(\langle z_{s,t}^N(x),\xi_1\rangle, \langle z_{s,t}^N(x),\xi_2\rangle, \cdots \langle z_{s,t}^N(x),\xi_m\rangle).$$

Since $|h^{(\mu)}(x)| \leq C_{51}\exp(\sum_{i=1}^m \sqrt{|x_i|})$, (4.11) of Lemma 6 yields that (4.18)

is dominated by

$$(4.19) \quad \sup_Q (1 + \sum_{i=1}^m \langle x,\xi_i\rangle^2 + \sum_{j=1}^\ell \langle x,\zeta_j\rangle^2)^n \exp(-\sum_{j=1}^\ell \sqrt{|\langle x,\zeta_j\rangle|})E[(\overline{\varphi}^{(v)}(z_{s,t}^N(x)))^4]^{1/4}$$

$$\leq C_{52} \sup_Q (1 + \sum_{i=1}^m \langle x,\xi_i\rangle^2 + \sum_{j=1}^\ell \langle x,\zeta_j\rangle^2)^n \exp(-\sum_{j=1}^\ell \sqrt{|\langle x,\zeta_j\rangle|})$$

$$\times E\left[\frac{(1 + \sum_{i=1}^m \langle z_{s,t}^N(x),\xi_i\rangle^2)^{4n}}{(1 + \sum_{i=1}^m \langle z_{s,t}^N(x),\xi_i\rangle^2)^{4n}}|\overline{\varphi}^{(v)}(z_{s,t}^N(x))|^4\right]^{1/4}$$

$$\leq C_{53}\|\varphi\|_n \sup_Q (1 + \sum_{i=1}^m \langle x,\xi_i\rangle^2 + \sum_{j=1}^\ell \langle x,\zeta_j\rangle^2)^n \exp(-\sum_{j=1}^\ell \sqrt{|\langle x,\zeta_j\rangle|})$$

$$\times E \left[\frac{1}{(1+ \sum_{i=1}^{m} \langle z_{s,t}^N(x), \xi_i \rangle^2)^{4n}} \right]^{1/4}.$$

where $\|\varphi\|_n = \sup_{\substack{x \in \mathbb{R}^m \\ 0 \leq r \leq n}} (1+|x|^2)^n |\varphi^{(r)}(x)|.$

On the other hand, we can verify the following lemma.

Lemma 7. For any $\xi_1, \xi_2, \cdots, \xi_m \in E$ and any integer $p \geq 1$, we have

$$E \left[\frac{1}{(1+ \sum_{i=1}^{m} \langle z_{s,t}^N(x), \xi_i \rangle^2)^p} \right] \leq C_{54}(T) \frac{1}{(1+ \sum_{i=1}^{m} \langle x, \xi_i \rangle^2)^p}, \quad 0 \leq s, t \leq T.$$

Proof. Setting $\theta(x) = \dfrac{1}{(1+ \sum_{i=1}^{m} \langle x, \xi_i \rangle^2)^p}$ and applying the Itô formula for

$\theta(z_{s,t}^N(x))$, we get

$$(4.20) \qquad E \left[\frac{1}{(1+ \sum_{i=1}^{m} \langle z_{s,t}^N(x), \xi_i \rangle^2)^p} \right] = \frac{1}{(1+ \sum_{i=1}^{m} \langle x, \xi_i \rangle^2)^p}$$

$$+ E \left[\int_s^t -2p(1+ \sum_{i=1}^{m} \langle z_{s,r}^N(x), \xi_i \rangle^2)^{-(p+1)} (\sum_{i=1}^{m} \langle z_{s,r}^N(x), \xi_i \rangle \langle \tilde{B}(r, z_{s,r}^{N-1}(x)), \xi_i \rangle) \, dr \right]$$

$$+ E \left[\int_s^t \sum_{j=1}^{\infty} \{ 2p(p+1)(1+ \sum_{i=1}^{m} \langle z_{s,r}^N(x), \xi_i \rangle^2)^{-(p+2)} (\sum_{i=1}^{m} \langle z_{s,r}^N(x), \xi_i \rangle \langle \tilde{A}(r, z_{s,r}^{N-1}(x))h_j^{(0)}, \xi_i \rangle)^2 \right.$$

$$\left. - p(1 + \sum_{i=1}^{m} \langle z_{s,r}^N(x), \xi_i \rangle^2)^{-(p+1)} (\sum_{i=1}^{m} \langle \tilde{A}(r, z_{s,r}^{N-1}(x))h_j^{(0)}, \xi_i \rangle^2) \} dr \right].$$

By the boundedness of $\tilde{A}_k(t,x)$, (4.20) is dominated by

$$\frac{1}{(1+ \sum_{i=1}^{m} \langle z_{s,r}^N(x), \xi_i \rangle^2)^p} + C_{55} \int_s^t E \left[\frac{1}{(1+ \sum_{i=1}^{m} \langle z_{s,r}^N(x), \xi_i \rangle^2)^p} \right] dr.$$

which yields the proof of the lemma, together with the Gronwall lemma.

Using this lemma, we see that the right hand side of (4.18) is dominated by

$$C_{56} \|\varphi\|_n \sup_Q (1+ \sum_{i=1}^m \langle x, \xi_i \rangle^2 + \sum_{j=1}^\ell \langle x, \zeta_j \rangle^2)^n \exp(- \sum_{j=1}^\ell \sqrt{|\langle x, \zeta_j \rangle|})$$

$$\times \frac{1}{(1+ \sum_{i=1}^m \langle x, \xi_i \rangle^2)^n} < \infty.$$

Hence $E[\tilde{F}(z_{s,t}^N(x))]$ is a weighted Schwartz functional and the proof of Proposition 2 is complete.

The following remark is immediate.

Remark. Under the assumptions of Proposition 2, $(L(t)F)(x) \in \mathfrak{D}_E$, if $F \in \mathfrak{D}_E$. .

§5. Theorem

Propositions 1 and 2, together with Remark 1, yield

Theorem. Suppose that the coefficients $A(t,x)$ and $B(t,x)$ satisfy the conditions (H1)-(H3) and are approximated by bounded smooth functionals on E'. Then $L(t)$ generates the Kolmogorov evolution operator $U(t,s)$ from $\mathfrak{D}_E$ into itself. Further under the same assumption on the initial value as in Proposition 1, the continuous $\mathfrak{D}(\mathfrak{D}_E')$-process solution of (1.1) is uniquely given by

$$X_F(t) = X_{U(t,0)F}(0) + W_F(t) + \int_0^t W_{L(s)U(t,s)F}(s)ds.$$

As a direct application of our theorem, we give below another approach to

the fluctuation problem in [4].

Example. Lattice system of interacting diffusions

First we begin to explain the system that Deuschel considered in [4]. Let Z^d be the d-dimensional lattice, $i=(i_1, i_2, \cdots, i_d) \in Z^d$ and $\mathscr{Y} = \mathscr{Y}(Z^d)$ the Schwartz space of rapidly decreasing sequences $\xi = (\xi_i)$, metrized by the countably many semi-norms:

$$\|\xi\|_p^2 = \sum_{i \in Z^d} (1+|i|)^{2p} |\xi_i|^2, \quad p=0,1,2,\cdots .$$

The dual space $\mathscr{Y}' = \mathscr{Y}'(Z^d)$ of $\mathscr{Y}$ is the collection of all slowly increasing sequences $x = (x_i)$ such that for some integer $p \geq 0$,

$$\|x\|_{-p}^2 = \sum_{i \in Z^d}(1+|i|)^{-2p}|x_i|^2 < \infty.$$

Let $b_i(x)$, $i \in Z^d$, be a real valued infinitely many times $\mathscr{Y}'_p$-Fréchet differentiable mapping on $\mathscr{Y}'$ for every integer $p \geq 0$ such that $b_i(x) = \hat{b}(\theta_i x)$, where $\hat{b}(x)$ is a real valued mapping on $\mathscr{Y}'$ and $\theta_i x = (x_{j+i})$.

(V1) We have some natural number p_0 such that

$$\sum_{i \in Z^d}^{\infty} (1+|i|)^{-2p_0}(\sup_{x \in \mathscr{Y}'} |b_i(x)|)^2 < \infty.$$

(V2) For any integers $n \geq 1$ and $p \geq 0$,

$$\sum_{i \in Z^d}^{\infty}(1+|i|)^{-2p}(\sup_{x \in \mathscr{Y}'} \|D^n b_i(x)\|_{H.S.}^{(p)})^2 < \infty.$$

(V3) For any integers $p \geq p_0$, $q \geq 0$ and $n \geq 0$, there exists a sequence of real valued bounded smooth functionals $b_i^{(m)}(x)$ such that

$$\lim_{m \to \infty} \sup_{x \in \mathscr{Y}'_p} \|D^n b_i(x) - D^n b_i^{(m)}(x)\|_{H.S.}^{(q)} = 0.$$

Let $x(t) = (x_i(t), i \in Z^d)$ be an $\mathscr{S}'(Z^d)$-valued solution of the following equation:

(5.1) $$x_i(t) = \sigma_i + B_i(t) + \int_0^t b_i(x(s))ds,$$

$$b_i(x) = \hat{b}(\theta_i x), \quad \theta_i x = (x_{j+i}),$$

where $(B_i(t))$ are independent copies of the 1-dimensional standard Brownian motion $B(t)$, (σ_i) are independent copies of the 1-dimensional random variable σ independent of $B(t)$ and for any $\epsilon > 0$, $E[\exp(\epsilon\|(\sigma_i)\|_{-p_0})] < \infty$. For a finite lattice $V \in Z^d$, consider

$$T_V(t) = |V|^{-1/2} \sum_{i \in V} \delta_{\theta_i x(t)}.$$

Now put

$$\langle U_V(t), \Phi \rangle = \langle T_V(t), \Phi \rangle - E[\langle T_V(t), \Phi \rangle], \quad \Phi \in C_0^\infty(\mathscr{S}').$$

Then it can be proved (see [18], [22]) that $U_V(t)$ becomes a strongly continuous $C_0^\infty(\mathscr{S}')'$-valued stochastic process. We will prove tightness for $U_V(t)$, $V \in Z^d$ following [5], [19], in $C([0,\infty); C_0^\infty(\mathscr{S}')')$. Let L_0 be an operator defined by

$$(L_0 F)(x) = \frac{1}{2} \text{trace}_{\ell^2(Z^d)} D^2 F(x) + DF(x)(b(x)), \quad F \in \mathfrak{D}_{\mathscr{S}'(Z^d)}.$$

where $b(x) = (b_i(x))$.

By the conditions (V1) and (V2), equation (5.1) is solved in $\mathscr{S}'_{p_0}$, so that $x(t) \in \mathscr{S}'_{p_0}$. Then we have by the exponential integrability proved in the same way as in Lemma 5,

$$E[\langle T_V(t), \Phi \rangle^2] \le C_{57} \|\Phi\|^2_{p_0, 0, 0}.$$

Since $C_0^\infty(\mathscr{Y}')$ is dense in $\mathscr{D}_{\mathscr{Y}'(\mathbf{Z}^d)}$, $T_V(t)$ is extended to a continuous $\mathscr{L}(\mathscr{D}_{\mathscr{Y}'(\mathbf{Z}^d)})$-process. We denote the extension by $T_{\phi,V}(t)$.

Let $\Phi(x) = \phi(\langle x,\xi_1\rangle, \langle x,\xi_2\rangle, \cdots, \langle x,\xi_n\rangle)$, $\phi \in S(\mathbb{R}^n)$. By the Itô formula, we get

$$(5.2) \qquad \langle T_V(t),\Phi\rangle - \langle T_V(0),\Phi\rangle = M_{\phi,V}(t) + \int_0^t T_{L_0\Phi,V}(s)ds,$$

where

$$M_{\phi,V}(t) = |V|^{-1/2} \sum_{i \in V} \int_0^t \sum_{j=1}^n \frac{\partial}{\partial x_j} \phi(\langle \theta_i x(s),\xi_1\rangle, \langle \theta_i x(s),\xi_2\rangle, \cdots, \langle \theta_i x(s),\xi_n\rangle)$$

$$\left(\sum_{k \in \mathbf{Z}^d} \xi_k^i \, dB(s) \right).$$

where $\xi_i = (\xi_k^i)$, $k \in \mathbf{Z}^d$.

From the independence of $B_i(t)$, $i \in V$ and the fact that $x(t) \in \mathscr{Y}'_{P_0}$, we have for $t \in [0,T]$,

$$(5.3) \qquad E[M_{\phi,V}(t)^4] \leq C_{58}\|\phi\|_{P_0,1,1}^4.$$

Then $M_{\phi,V}(t)$ can be extended to a continuous $\mathscr{L}(\mathscr{D}_{\mathscr{Y}'(\mathbf{Z}^d)})$-process and has the same regularity properties that the $\mathscr{L}(\mathscr{D}_{\mathscr{Y}'(\mathbf{Z}^d)})$-Wiener process has. Conditions (V1)-(V3) guarantee that L_0 belongs to the class dealt with in the Theorem. We use the same notation $U(t,s)$ to represent the evolution operator generated by L_0. Thus the solution of (5.2) is given as follows:

$$\langle T_V(t),\Phi\rangle = T_{U(t,0)\Phi,V}(0) + M_{\phi,V}(t) + \int_0^t M_{L_0 U(t,s)\Phi,V}(s)ds$$

as in the proof of Proposition 1. Hence by (5.3) and the Kolmogorov test for a real Wiener process, we get

$$E[|\langle U_V(t)-U_V(s),\Phi\rangle|^4] \leq C_{59}|t-s|^2$$

and further

$$E[|\langle U_V(t),\Phi\rangle|^2] \le C_{60}\{\|\Phi\|^2_{p_0,1,1} + \sup_{0\le s\le t} \|U(t,s)\Phi\|^2_{p_0,1,3}\}$$

which proves the tightness in $C([0,\infty);C_0^\infty(\mathcal{Y}')')$, ([5], [18]). By the Skorokhod theorem and the usual limiting argument, the limit process $N(t)$ of $U_V(t)$ satisfies the SDE

$$(5.4) \qquad \langle N(t)-N(0),\Phi\rangle = W_\Phi(t) + \int_0^t N_{L_0\Phi}(s)ds,$$

where $N_F(t)$, $F \in \mathcal{D}_{\mathcal{Y}'(\mathbb{Z}^d)}$, is the extension of $N(t)$ and $W_F(t)$ is a $\mathcal{L}(\mathcal{D}_{\mathcal{Y}'(\mathbb{Z}^d)})$-Wiener process [8].

The uniqueness for solutions of the equation (5.4) discussed in Theorem implies the identification of the distribution of the limit process, ([20], [21]), which implies that $U_V(t)$ converges to a Gaussian field in $C([0,\infty);C_0^\infty(\mathcal{Y}')')$.

§6. A fluctuation theorem for a system of interacting, spatially distributed neurons.

A problem in neurophysiology that has received considerable attention in recent years, is the stochastic behavior of the voltage potential of a spatially distributed neuron [11,26]. When the spatial dimension of the neuronal membrane is greater than one, the voltage potential is modeled as a stochastic process taking values in the dual of some nuclear space such as the space of Schwartz distributions $\mathcal{Y}'(\mathbb{R}^d)$. The SDE satisfied by the voltage potential is best introduced via the following general model: Let H be a real separable Hilbert space. In applications, usually $H=L^2(\mathfrak{X},d\mu)$ where $\mathfrak{X}$ is the membrane of the spatially extended neuron (e.g. $\mathfrak{X} = [0,b]$, a d-dimensional rectangle or a compact Riemanian manifold with or without boundary, and μ is

the appropriate natural measure on $\mathfrak{X}$). Let T_t be a strongly continuous semigroup on H generated by a closed, densely defined operator $\mathfrak{X}$ such that $(\mathfrak{X}\xi,\xi)_H \leq 0$ for $\xi \in Dom(\mathfrak{X})$ where $(\cdot,\cdot)_H$ denotes the inner product of H. Assume that some power of the resolvent of $\mathfrak{X}$ is a Hilbert-Schmidt operator i.e.

$$(6.1) \qquad (\lambda I - \mathfrak{X})^{-r_1} \text{ is Hilbert-Schmidt for some } r_1 > 0.$$

Then there is a CONS $\{\varphi_j\}_{j \geq 1}$ in H such that $-\mathfrak{X}\varphi_j = \lambda_j \varphi_j$ for any $j \geq 1$ and $0 \leq \lambda_1 < \lambda_2 < \ldots$. Set

$$E = \{\xi \in H; \sum_{j=1}^{\infty} (1+\lambda_j)^{2r}(\xi,\varphi_j)_H^2 < \infty \text{ for any } r \geq 0\}.$$

Define the inner product on E,

$$(\xi,\zeta)_r = \sum_{j=1}^{\infty} (1+\lambda_j)^{2r}(\xi,\varphi_j)_H(\zeta,\varphi_j)_H$$

and E_r as the $\|\cdot\|_r$-completion of E, ($\|\xi\|_r^2 = (\xi,\xi)_r$) and E_r' as the dual of the Hilbert space E_r. For $r < s$, $E_s \subseteq E_r$ and $E_0 = H$. Condition (6.1) implies that the canonical injection $E_p \to E_r$ is Hilbert-Schmidt if $p > r+r_1$. Hence E is nucelar.

Since $\mathfrak{X}$ generates T_t on H, we have for $\xi \in E$ and $t > 0$,

$$T_t\xi = \sum_{j=1}^{\infty} e^{-t\lambda_j}(\xi,\varphi_j)_0\varphi_j.$$

The following properties of T_t can be easily verified:

(a) $T_t E \subset E$;

(b)   The restriction of T_t to E is an E-continuous semigroup;

(c) $t \to T_t\xi$ is continuous for every $\xi \in E$;

(d)   The restriction of $\mathfrak{X}$ on E maps E into E and is the generator of the

semigroup T_t on E;

(e)   For for any $\xi \in E$ and $t > 0$,

$$\|T_t\xi\|_r \leq \|\xi\|_r .$$

The voltage potential is then derived as the solution of an E'-valued SDE

$$(6.2) \qquad\qquad dX(t) = d\beta(t) + \mathcal{X}'X(t)dt$$

where $\mathcal{X}'$ is the adjoint of $\mathcal{X}$ on E and $\beta(t)$ is an E'-valued Wiener process with
$E[\langle\beta(t),\xi\rangle \langle\beta(s),\zeta\rangle] = (t \wedge s)Q(\xi,\zeta)$, Q being an E-continuous quadratic form.

Let us now define

$$\langle V(t)x,\xi\rangle = \langle x,T_t\xi\rangle \qquad \forall x \in E', \ \xi \in E.$$

Then, using property (e) above we have

$$\|V(t)x\|_{-r} = \sup_{\|\xi\|_r \leq 1} |\langle x,T_t\xi\rangle| \leq \|x\|_{-r} \sup_{\|\xi\|_r \leq 1} \|T_t\xi\|_r \leq \|x\|_{-r}$$

and so

$$(6.3) \qquad\qquad \sup_{0 \leq t \leq T} \|V(t)x\|_{-r} \leq \|x\|_{-r} .$$

(6.3) is a special case of condition (VI2) below, which is thus satisfied for
the class of spatially extended neurons whose voltage potentials are modeled by
(6.2). For specific examples of $L^2(\mathcal{X},d\mu)$ and the semigroup T_t which describes
the deterministic part of the behavior of the neuron, see [11].

We now come to the question of interacting assemblies of a very large
number of neurons. This appears to be a very important problem of
physiological interest since such large systems are involved in the functioning
of the central nervous system. The difficulty consists in discovering the
precise nature of the interaction in a mathematical form. In this section we
consider an interaction similar to the mean-field interaction in particle

diffusions. Another, possibly more realistic interaction known in the physiological literature as "parallel fiber interaction" will be investigated in our future work.

Let $b(x,y)$ be a mapping from $E' \times E'$ to some E'_{p_6} such that $b(\cdot,\cdot)$ is infinitely many E'_p-Fréchet differentiable for every integer $p \geq 0$ and with all derivatives bounded;

(VI1)
$$\sup_{x,y \in E'} \|D_x^k D_y^m b(x,y)\|_{H.S.}^{(p)} < \infty$$

for any integers k, m and $p \geq 0$. Here D_x and D_y denote the Fréchet derivatives with respect to variables x and y. The i-th component $X_i^{(n)}(t)$ of the n-system of diffusions is obeyed by the following stochastic differential equation:

$$(6.4) \quad dX_i^{(n)}(t) = d\beta_i(t) + \{\mathcal{A}(t)X_i^{(n)}(t) - \frac{1}{n}\sum_{j=1}^{n} b(X_i^{(n)}(t), X_j^{(n)}(t))\}dt,$$
$$i=1,2,\cdots,n,$$

where $\{\beta_i(t)\}$ are independent copies of an E'-valued Wiener process $\beta(t)$.

Suppose that $\mathcal{A}(t)$ generates the strongly continuous contraction evolution operator $V(t,s)$ from E' to itself such that for any integer p and any $T>0$, there exists some integer $n(p,T) \geq p$ satisfying

$$(VI2) \qquad \sup_{0 \leq s \leq t \leq T} \|V(t,s)x\|_{-n(p,t)} \leq \|x\|_{-p}.$$

Without loss of generality, we may assume $n(p,T) \leq n(q,T)$ if $p \leq q$. Then (6.4) is equivalent to

$$(6.5) \quad X_i^{(n)}(t) = V(t,0)\sigma_i + \int_0^t V(t,s)d\beta_i(s) + \int_0^t V(t,s)(\frac{1}{n}\sum_{j=1}^{n} b(X_i^{(n)}(s),X_j^{(n)}(s)))ds.$$

We assume the initial value σ_i to be an independent copy of σ such that

$E[\exp(\epsilon\|\sigma\|_{-p_7})] < \infty$ for every $\epsilon > 0$ and some natural number p_7.

Since $Q(\xi,\zeta)$ is a continuous quadratic form on E, there exists an integer r such that

$$Q(\xi,\zeta) = (Q_r^{1/2}\xi, Q_r^{1/2}\zeta)_r$$

where Q_r is a self-adjoint operator on E_r. Then clearly, $\beta(t) \in E'_{p_8}$ for some integer $p_8 > r$.

The solution of (6.5) until time T is easily obtained by the usual method of successive approximations in $E'_{n(p_9,T)}$, where $p_9 = p_6 \vee p_7 \vee p_8$.

For the finite measure $v(dx)$ on E', set $b[x,v] = \int_{E'} b(x,y)v(dy)$, where the integral is the Bochner integral on E' and consider

(6.6)
$$dX_1(t) = d\beta_1(t) + \{\mathcal{A}(t)X_1(t) + b[X_1(t),u]\}dt,$$
$$u(t,dx) = \text{the distribution of } X_1(t).$$

Then according to the following lemma the empirical distribution $\frac{1}{n}\sum_{j=1}^{n}\delta_{X_j^{(n)}(t)}$ converges to $u(t,dx)$ in probability in the usual weak convergence of measures, where δ_x is the Dirac measure at x in E'.

Lemma 8. For any $T > 0$ and integer $j \geq 1$,
$$E[\|X_1^{(m)}(t) - X_1(t)\|_{-n(p_6,T)}^{2j}] \leq C_{61}(T)/m^j, \qquad 0 \leq t \leq T.$$

Proof. Put $n_0 = n(p_6,T)$. Then the condition (VI1) yields

$$\|b(X_1^{(m)}(t), X_j^{(m)}(t)) - b(X_1(t), X_j^{(m)}(t))\|_{-p_6}$$

$$\leq \sup_{x,y \in E'} \|D_x b(x,y)\|_{H.S.}^{(n_0)} \|X_1^{(m)}(t) - X_1(t)\|_{-n_0}$$

$$\leq C_{62}\|X_i^{(m)}(t) - X_i(t)\|_{-n_0}$$

and

$$\|b(X_i(t),X_j^{(m)}(t)) - b(X_i(t),X_j(t))\|_{-p_6} \leq C_{63}\|X_j^{(m)}(t) - X_j(t)\|_{-n_0}.$$

so that we have

(6.7)
$$E[\|X_i^{(m)}(t) - X_i(t)\|_{-n_0}^{2j}]$$

$$\leq C_{64}(T)\int_0^t E[\|V(t,s)\{\frac{1}{m}\sum_{j=1}^m b(X_i^{(m)}(s),X_j^{(m)}(s)) - b[X_i(s),u]\}\|_{-n_0}^{2j}]ds$$

$$\leq C_{65}(T)\int_0^t \{E[\|X_i^{(m)}(s) - X_i(s)\|_{-n_0}^{2j}] + \frac{1}{m}\sum_{j=1}^m E[\|X_j^{(m)}(s) - X_j(s)\|_{-n_0}^{2j}]$$

$$+ E[\|\frac{1}{m}\sum_{j=1}^m \{b(X_i(s),X_j(s)) - b[X_i(s),u]\}\|_{-p_6}^{2j}]\}ds.$$

Noticing the independence of $X_i(t)$, $i=1,2,\cdots,m$ and the condition (VI1), we have

(6.8)
$$E[\|\frac{1}{m}\sum_{j=1}^m \{b(X_i(s),X_j(s))-b[X_i(s),u]\}\|_{-p_6}^{2j}] \leq C_{66}(T)/m^j.$$

Therefore Gronwall's inequality, together with (6.7) and (6.8), implies the assertion of Lemma 8.

Now we proceed to the discussion of the fluctuation problem. We are able to consider $U_n(t) = \sqrt{n}\left(\frac{1}{n}\sum_{j=1}^n \delta_{X_j^{(n)}(t)} - u(t,dx)\right)$ as a $C_0^\infty(E')'$-valued continuous stochastic process [18], [22]. To check the tightness of $U_n(t)$ in $C([0,\infty);$ $C_0^\infty(E')')$ of all continuous mappings from $[0,\infty)$ into $C_0^\infty(E')'$, it is enough to verify the Kolmogorov tightness criterion for $\langle U_n(t),\Phi\rangle$, $\Phi \in C_0^\infty(E')$, where $\langle \cdot , \rangle$ denotes the canonical bilinear form on $C_0^\infty(E')' \times C_0^\infty(E')$. [5], [19].

We have the following exponential integrability.

Lemma 9. For any $\alpha > 0$, $T > 0$ and any integer $p \geq n(p_9, T)$, there exists a constant $C_{67} = C_{67}(\alpha, T, p)$ such that

$$\sup_{0 \leq t \leq T} E[e^{\alpha \|X_i^{(n)}(t)\|_{-p}}] \vee E[e^{\alpha \|X_i(t)\|_{-p}}] \leq C_{67}.$$

Proof. Set $n_0 = n(p_9, T)$. Assumptions (VI1) and (VI2) give

$$\max\{\|X_i^{(n)}(t)\|_{-n_0}, \|X_i(t)\|_{-n_0}\} \leq \|\sigma_i\|_{-p_7} + C_{68} + \|\int_0^t V(t,s)dB_i(s)\|_{-n_0}$$

and hence the lemma can be proved in the same way as Lemma 5.

Once we know Lemmas 8 and 9, we can check the moment condition:

$$(6.9) \qquad E[|\langle U_n(t) - U_n(s), \Phi \rangle|^4] \leq C_{69}(\Phi)|t-s|^2.$$

(see [8]). Similarly we have

$$(6.10) \qquad \sup_{0 \leq t \leq T} E[\langle U_n(t), \Phi \rangle^2] \leq C_{70}(T) \|\Phi\|^2_{n(p_9, T), n(p_9, T), 1}.$$

Then a subsequence of $U_n(t)$ converges to $U(t)$ in $C([0, \infty); C_0^\infty(E')')$. Further (6.9) and (6.10) guarantee that $U_n(t)$ and $U(t)$ can be extended to continuous $\mathcal{L}(\mathcal{D}_{E'})$-processes and so we denote the extensions by $(U_n)_F(t)$ and $U_F(t)$. $F \in \mathcal{D}_{E'}$, respectively.

For any $F \in \mathcal{D}_{E'}$, define

$$(K(t)F)(x) = \frac{1}{2} \text{trace}_{E_r} D^2 F(x) \circ [(Q_r^{1/2})^* \times (Q_r^{1/2})^*] + DF(x)(b[x, u] + \mathcal{A}(t)x)$$

$$+ \int_{E'} DF(y)(b(y, x))u(t, dy)$$

and

$$W_F(t) = U_F(t) - U_F(0) - \int_0^t U_{K(s)F}(s)ds.$$

where trace $_{E_r} D^2F(x) \circ [(Q_r^{1/2})^* \times (Q_r^{1/2})^*] = \sum_{j=1}^{\infty} D^2F(x)((Q_r^{1/2})^* h_j^{(r)}, (Q_r^{1/2})^* h_j^{(r)})$

and $*$ means the adjoint operator with respect to the dual pair on $E' \times E$.

By following the argument of [8] word by word, we have the proof that $W_F(t)$ is a continuous $\mathcal{L}(\mathcal{D}_{E'})$-Wiener process. Thus any limit process of convergent subsequences of $U_n(t)$ satisfies the weak SDE of type (1.1).

Now we impose a rather technical condition on $b(x,y)$.

(VI3) For any $\epsilon > 0$ and any integers $p, q, n \geq 0$, there exists a C_b^{∞}-function $\bar{b}(x,y)$ of $\mathbb{R}^m \times \mathbb{R}^{m'}$ to E'_{p_6} such that

$$\sup_{\substack{x \in E'_p}} \| D_x^{\mu} D_y^{\nu}[b(x,y) - \bar{b}(\langle x, \xi_1 \rangle, \langle x, \xi_2 \rangle, \cdots \langle x, \xi_m \rangle,$$

$$\langle y, \zeta_1 \rangle, \langle y, \zeta_2 \rangle, \cdots, \langle y, \zeta_{m'} \rangle)] \|_{H.S.}^{(q)} < \epsilon,$$

$$0 \leq \mu + \nu \leq n, \quad \xi_i, \zeta_j \in E, \quad i = 1, 2, \cdots, m \text{ and } j = 1, 2, \cdots, m'.$$

Here C_b^{∞}-function means $\bar{b}(x,y)$ itself and all the derivatives are bounded.
We set

$$(A(t)F)(x) = \frac{1}{2} \text{trace}_{E_r} D^2F(x) \circ [(Q_r^{1/2})^* \times (Q_r^{1/2})^*] + DF(x)(b[x,u] + \mathcal{A}(t)x)$$

and

$$(J(t)F)(x) = \int_E DF(y)(b(y,x))u(t,dy).$$

Though $\mathcal{A}(t)x$ is not bounded, from a part of the proof of Proposition 2 and the assumptions (VI1) and (VI3), we can show

$$A(t)\mathcal{D}_{E'} \subset \mathcal{D}_{E'} \quad \text{and} \quad J(t)\mathcal{D}_{E'} \subset \mathcal{D}_{E'}.$$

Since

$$\eta_{s,t}(x) = V(t,s)x + \int_s^t V(t,r)d\beta(r) + \int_s^t V(t,r)b[\eta_{s,r}(x),u]dr,$$

choosing $q' > n(q'',T)$ such that $q'' > \max\{p, p_9, q\}$ and $\sum_{j=1}^{\infty} \|h_j^{(q)}\|_{-q''}^2 < \infty$ in the proofs of Propositions 1 and 2 and recalling the condition (VI2), we conclude

that Lemma 5 holds if q' in the right hand side is replaced by q" and hence, together with (VI1) and (VI3), we obtain that $A(t)$ generates the Kolmogorov evolution operator from $\mathfrak{D}_E$, into itself similarly. Further since $J(t)$ satisfies the condition of Proposition 2 in [21] and the proof of Proposition 2 in [21] is valid for any Fréchet space, $K(t) = A(t)+J(t)$ generates the Kolmogorov evolution operator $U(t,s)$ from $\mathfrak{D}_E$, into itself. Since the Theorem gives the identification of the distributions of the limit processes $U(t)$, we obtain the conclusion that under the assumptions (VI1) – (VI3) and the exponential integrability of σ, $U_n(t)$ converges to a Gaussian field governed by the weak SDE of type (1.1) in $C([0,\infty); C_0^\infty(E')')$, namely,

$$dX_F(t) = dW_F(t) + X_{K(t)F}(t)dt,$$

where $W_F(t)$ is an $\mathcal{L}(\mathfrak{D}_E,)$-Wiener process with

$$E[W_F(t)W_G(s)] = \int_0^{t\wedge s}(\int_{E'} \sum_{j=1}^{\infty} DF(x)((Q_r^{1/2})^* h_j^{(r)})DG(x)((Q_r^{1/2})^* h_j^{(r)})u(\tau,dx))d\tau$$

and $X_F(0)$ is a Gaussian random variable with

$$E[X_F(0)X_G(0)] = E[F(\sigma)G(\sigma)] - E[F(\sigma)]E[G(\sigma)], \quad F,G \in \mathfrak{D}_E,.$$

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